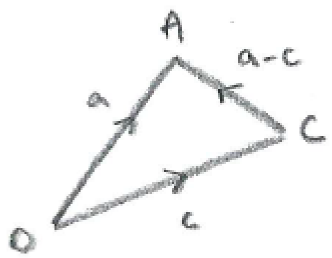
STEP 2005, Paper 3, Q8 - Solution (4 pages; 12/5/18)

$|a-c|^2 = (a-c)(a-c)^* = (a-c)(a^*-c^*)$

$= aa^* + cc^* - ac^* - ca^*$, as required (1)



By the Cosine rule, $|c|^2 = |a|^2 + |a-c|^2 - 2|a||a-c|cosA$

$A = 90° \Leftrightarrow cosA = 0$ (as $A < 180°$)

$\Leftrightarrow |c|^2 = |a|^2 + |a-c|^2$, since $|a| \neq 0$ & $|a-c| \neq 0$ $(as\ a \neq c)$

[this justification is necessary in order for the ⇐ part of the statement to be established]

[Alternatively, just use the fact that Pythagoras' theorem holds if and only if the triangle is right-angled - as per the Hints & Answers.]

$\Leftrightarrow$ from (1) that $|c|^2 - |a|^2 - (aa^* + cc^* - ac^* - ca^*) = 0$

$\Leftrightarrow |c|^2 - |a|^2 - (|a|^2 + |c|^2 - ac^* - ca^*) = 0$

$\Leftrightarrow -2|a|^2 + ac^* + ca^* = 0$

$\Leftrightarrow -2|a|^2 + ac^* + ca^* = 0$

$ac^* + ca^* = 2|a|^2 = 2aa^*$, as required

[being careful not to seem to jump to the answer given]

P lies on the circle $\Leftrightarrow |ab-c|^2 = |a-c|^2$ (1)

P' lies on the circle $\Leftrightarrow |\frac{a}{b^*} - c|^2 = |a-c|^2$ (2)

and, as OA (extended) is a tangent to the circle, it follows that OA is perpendicular to AC, so that $A = 90°$ and hence $2aa^* = ac^* + ca^*$ (3),

from the previous result

[It is also worth looking ahead to the last part of the question, in case this could influence how we approach the present part.]

From the initial result,

(1) $\Leftrightarrow ab(ab)^* + cc^* - abc^* - c(ab)^* = aa^* + cc^* - ac^* - ca^*$

and (2) $\Leftrightarrow \frac{a}{b^*}\left(\frac{a}{b^*}\right)^* + cc^* - \left(\frac{a}{b^*}\right)c^* - c\left(\frac{a}{b^*}\right)^* = aa^* + cc^* - ac^* - ca^*$

Noting that $\left(\frac{a}{b^*}\right)^* = \frac{a^*}{b}$, cancelling the cc^* terms and multiplying by bb^*, this $\Leftrightarrow aa^* - ac^*b - ca^*b^* = aa^*bb^* - ac^*bb^* - ca^*bb^*$

$\Leftrightarrow aa^* - ac^*b - ca^*b^* - aa^*bb^* + ac^*bb^* + ca^*bb^* = 0$ (2′)

[the last step is intended to make it easier to compare with (1)]

Then (1) $\Leftrightarrow aa^* - ac^* - ca^* - aa^*bb^* + abc^* + ca^*b^* = 0$ (1′)

We could replace $aa^* - ac^* - ca^*$ with $-aa^*$, from (3), but noting that the last part of the question involves (3), it may be best to write

$E = 2aa^* - ac^* - ca^*$ for the time being (where (3) $\Leftrightarrow E = 0$)

Thus (1′) $\Leftrightarrow E - aa^* - aa^*bb^* + abc^* + ca^*b^* = 0$ (1″)

[We want to show that (2′) $\Leftrightarrow$ (1″). One technique is to force (1″) into the form of (2′):]

Then $(1'') \Leftrightarrow -E + aa^* + aa^*bb^* - abc^* - ca^*b^* = 0$

and considering $(2')$, this

$\Leftrightarrow (aa^* - ac^*b - ca^*b^* - aa^*bb^* + ac^*bb^* + ca^*bb^*)$

$+(2aa^*bb^* - ac^*bb^* - ca^*bb^*) - E = 0 \quad (1''')$

Writing $F = 2aa^*bb^* - ac^*bb^* - ca^*bb^*$,

we can then say that $(1''') \Leftrightarrow (2')$ if it can be shown that $F = 0$ (since

$(3) \Leftrightarrow E = 0)$

Now $F = (2aa^* - ac^* - ca^*)bb^* = Ebb^* = 0$

For the last part, if we write

$G = aa^* - ac^*b - ca^*b^* - aa^*bb^* + ac^*bb^* + ca^*bb^*$,

then $(1) \Leftrightarrow G + Ebb^* - E = 0$,

$(2) \Leftrightarrow G = 0$,

and $(3) \Leftrightarrow E = 0$

Then, if (1) and (2) hold, it follows that $Ebb^* - E = 0$,

ie $E(bb^* - 1) = 0$,

so that, since $bb^* \neq 1, E = 0$

ie $2aa^* - ac^* - ca^* = 0$,

from which it follows that $A = 90°$ and hence OA is a tangent to the circle

[Note how it's worth developing the various equations at the same time, in order to get ideas as to the best way to proceed.]