STEP 2005, Paper 3, Q3 - Solution (3 pages; 12/5/18)

$f\left(g(x)\right) = (x^2 + rx + s)^2 + p(x^2 + rx + s) + q$

$= x^4 + r^2x^2 + s^2 + 2rx^3 + 2sx^2 + 2rsx + px^2 + prx + ps + q$

$= x^4 + 2rx^3 + (r^2 + 2s + p)x^2 + (2rs + pr)x + s^2 + ps + q$

Equating coefficients,

$a = 2r$

$b = r^2 + 2s + p$

$c = 2rs + pr = r(2s + p) = \frac{a}{2}(b - \left(\frac{a}{2}\right)^2)$

$d = s^2 + ps + q$

$\Leftrightarrow c = \frac{a}{8}(4b - a^2)$ (1) (as q is just chosen to be $d - s^2 + ps$)

[Let A be the statement that (1) is true, and B the statement that the quartic can be written in the form $f\left(g(x)\right)$. We have shown that A and B are equivalent. The official solution seems to labour this a bit, by showing (as a sort of check) how $f\left(g(x)\right)$ is actually obtained, if A applies. In the above equations, $2s + p$ can be treated as a single parameter (b & c are both functions of $2s + p$; had they been independent functions of s & p, then it would just be a matter of solving simultaneous equations, in order to find the required values of s & p, and no condition would be necessary). In the official solution, p is set equal to 0, to simplify matters. However, it is doubtful whether it is worth going to this trouble for future questions: normally the examiners are happy if the equivalence symbol ⇔ is used (provided that the equivalence is clear). The one consolation with this often fiddly issue, is that, although a few marks may be dropped by omitting a full proof, it doesn't prevent progress with the rest of the question.]

[next part:]

$(x^2 + vx + w)^2 - k = x^4 + v^2x^2 + w^2 + 2vx^3 + 2wx^2 + 2vwx - k$

Equating coefficients, to give the following set of equations (2):

$a = 2v$

$b = v^2 + 2w$

$c = 2vw$

$d = w^2 - k$

(2) is equivalent to $b = \left(\frac{a}{2}\right)^2 + 2\left(\frac{c}{a}\right)$ for suitable u, v & w

$\Leftrightarrow 4ab = a^3 + 8c \Leftrightarrow c = \frac{a}{8}(4b - a^2)$, which is (1)

From the equation $x^4 - 4x^3 + 10x^2 - 12x + 4 = 0$, we can obtain suitable values for v, w & k , from (2), as follows:

Let $-4 = 2v$, $10 = v^2 + 2w$, $-12 = 2vw$ & $4 = w^2 - k$

Thus $v = -2$,

$10 = 4 + 2w \Rightarrow w = 3$ [and $-12 = 2(-2)(3)$]

and $k = 9 - 4 = 5$

So the equation becomes $(x^2 - 2x + 3)^2 - 5 = 0$

$\Rightarrow x^2 - 2x + 3 \pm \sqrt{5} = 0$

$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(3 \pm \sqrt{5})}}{2} = 1 \pm \sqrt{-2 \pm \sqrt{5}}$,

and hence the roots are $1 \pm \sqrt{\sqrt{5} - 2}$ and $1 \pm i\sqrt{\sqrt{5} + 2}$

[Because Further Maths knowledge is assumed for STEP 3, the roots in the last part should include complex ones.]