STEP 2005, Paper 3, Q2 - Solution (2 pages; 11/5/18)

[According to the Examiner's Report, this question was attempted by almost all candidates. There are 3 "show that" results, and the method is standard, making it a very promising question. The solution is not particularly long (and this is reasonably evident from the question).]

$\frac{dy}{dx} = -\frac{xy}{x^2+a^2} \Rightarrow 2\int \frac{1}{y}dy = -\int \frac{2x}{x^2+a^2}dx$

$\Rightarrow 2\ln|y| = -\ln(x^2+a^2) + \ln(A)$, where $A > 0$

$\Rightarrow ln(y^2) = ln\left(\frac{A}{x^2+a^2}\right)$

$\Rightarrow y^2 = \frac{A}{x^2+a^2}$

$\Rightarrow y^2(x^2+a^2) = c^2$ (as $A > 0$)

To sketch the curve:

(a) $x = 0 \Rightarrow y = \pm\frac{c}{a}; y = 0$ only occurs at $x = \pm\infty$

(b) The curve is defined for all values of x.

(c) As $x \to \pm\infty, y \to 0$

(d) To find the range of y:

$y^2 = \frac{c^2}{x^2+a^2}$, so that y^2 is maximised when $x = 0$; ie y lies in $[-\frac{c}{a}, \frac{c}{a}]$

(though $y \neq 0$)

(e) $\frac{dy}{dx} = 0 \Rightarrow x = 0$ *or* $y = 0$ (not possible, but $\frac{dy}{dx} \to 0$ as $y \to 0$)

(f) $\frac{dy}{dx} > 0$ when $x > 0$ & $y < 0$ *or* $x < 0$ & $y > 0$

$\frac{dy}{dx} < 0$ when $x > 0$ & $y > 0$ or $x < 0$ & $y < 0$

(g) Replacing $x\ with - x$ $or\ y\ with - y$ has no effect, so there is symmetry about both the x & $y\ axes$.

$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx} = 2x - \frac{2xy^2}{x^2+a^2} = 2x - \frac{2xc^2}{(x^2+a^2)^2}$

Then $\frac{d^2}{dx^2}(x^2 + y^2) = 2 - \frac{2c^2}{(x^2+a^2)^4}\{(x^2 + a^2)^2(1) - x(2)(x^2 + a^2)(2x)\}$

$= 2\left\{1 - \frac{c^2}{(x^2+a^2)^2}\right\} + \frac{8c^2x^2}{(x^2+a^2)^3}$, as required

(i) Distance from Origin $= x^2 + y^2$

To minimise this distance:

$\frac{d}{dx}(x^2 + y^2) = 0 \Rightarrow 2x\left\{1 - \frac{c^2}{(x^2+a^2)^2}\right\} = 0$

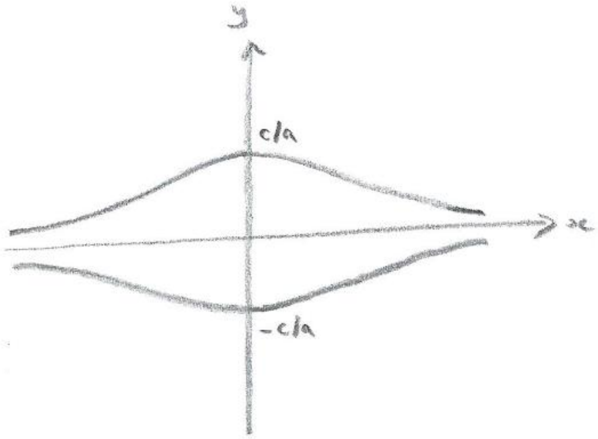
$\Rightarrow x = 0\ or\ x^2 + a^2 = c \quad (1)$

But as $c < a^2, x^2 + a^2 = c$ is not possible; so $x = 0$

$\frac{d^2}{dx^2}(x^2 + y^2)|(x = 0) \quad = 2\left\{1 - \frac{c^2}{a^4}\right\} > 0,$ as $c < a^2$

Thus the distance from the Origin is minimised when $x = 0$,

and $y^2a^2 = c^2$, so that $y = \pm\frac{c}{a}$, as required.



(ii) When $c > a^2$, from (1), either $x = 0$ *or* $x^2 + a^2 = c$

For $x = 0$:

$$c > a^2 \Rightarrow \frac{d^2}{dx^2}(x^2 + y^2)|(x = 0) < 0 \Rightarrow maximum$$

For $x^2 + a^2 = c$:

$$\frac{d^2}{dx^2}(x^2 + y^2)|(x^2 + a^2 = c) = 0 + \frac{8c^2(c-a^2)}{c^3} = \frac{8(c-a^2)}{c} > 0,$$

as $c > a^2 > 0$,

so that there is a minimum at $x = \pm\sqrt{c - a^2}$,

when $y^2 = c$, and hence the required points are $(\pm\sqrt{c - a^2}, \pm\sqrt{c})$