STEP 2005, Paper 1, Q14 – Solution (2 pages; 10/5/18)

[According to the ER, there were no successful attempts at this question; which indicates how under-rated these distribution questions are: there is hardly any theory involved, and the question often amounts to no more than a bit of integration.]

[Be careful not to use the cdf instead of the pdf, or vice-versa, for distribution questions.]

(i) Total prob. =1, so

$m + k(1 - e^{-\infty}) = 1 \Rightarrow m + k = 1 \Rightarrow k = 1 - m$

(ii) For $0 \leq X < \infty$, pdf of X $= \frac{d}{dx}\left(k(1 - e^{-x})\right)$

$= ke^{-x} = (1 - m)e^{-x}$

$E(X) = P(X = -1)(-1) + \int_0^\infty x.(1 - m)e^{-x}dx$ (*)

$= -m + (1 - m)[x(-e^{-x})]_0^\infty - (1 - m)\int_0^\infty (-e^{-x})dx$

[integrating by Parts]

since $xe^{-x} \to 0 \; as \; x \to \infty$

$= -m + 0 + (1 - m)[-e^{-x}]_0^\infty$

$= -m + (1 - m)(0 + 1) = 1 - 2m$

(iii) $Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = m(-1)^2 + \int_0^\infty x^2.(1 - m)e^{-x}dx$

$= m + (1-m)[x^2(-e^{-x})]_0^{\infty} - (1-m)\int_0^{\infty} 2x(-e^{-x})dx$

$= m + 0 + 2(1-m)\int_0^{\infty} xe^{-x}dx$

$= m + 2[E(X) + m],\ from\ (*)$

$= m + 2[1 - 2m + m] = 2 - m$

Then $Var(X) = 2 - m - (1-2m)^2 = 2 - m - 1 + 4m - 4m^2$

$= 1 + 3m - 4m^2$

Let M be the median.

Then $P(X < M) = \frac{1}{2}$,

so that $m + k(1 - e^{-M}) = 1/2$ (since $m < 1/2$, so that $M > 0$)

Then $1 - e^{-M} = \frac{\frac{1}{2} - m}{1-m} = \frac{1-2m}{2(1-m)}$

$\Rightarrow e^{-M} = 1 - \frac{1-2m}{2(1-m)} = \frac{2-2m-1+2m}{2(1-m)} = \frac{1}{2(1-m)}$

$\Rightarrow e^{M} = 2(1-m)$ and $M = \ln(2-2m)$

(iv) $E(|X|^{1/2}) = 1(m) + \int_0^{\infty} x^{1/2}(1-m)e^{-x}dx$

Let $y^2 = x$, so that $2ydy = dx$

Then $E(|X|^{1/2}) = m + 2(1-m)\int_0^{\infty} y^2 e^{-y^2}dy$

$= m + 2(1-m).\frac{1}{4}\sqrt{\pi}$

$= m + \frac{1}{2}(1-m)\sqrt{\pi}$