

Roots of Polynomial Equations - Exercises (Solutions)

(6 pages; 15/2/17)

(1) If the quadratic equation $2x^2 + 5x - 9 = 0$ has roots α and β , find the quadratic equation which has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

Solution

Method 1

$$\alpha + \beta = -\frac{5}{2} \text{ and } \alpha\beta = -\frac{9}{2}$$

Let the new equation be $x^2 + bx + c = 0$

$$\text{Then } \frac{1}{\alpha} + \frac{1}{\beta} = -b \text{ and } \frac{1}{\alpha} \cdot \frac{1}{\beta} = c ,$$

$$\text{so that } b = \frac{-(\alpha+\beta)}{\alpha\beta} = -\frac{5}{9} \text{ and } c = -\frac{2}{9}$$

$$\text{and the new equation is } x^2 - \frac{5x}{9} - \frac{2}{9} = 0$$

$$\text{or } 9x^2 - 5x - 2 = 0$$

[Note that, if written as $-9x^2 + 5x + 2 = 0$, then the coefficients of the original equation are reversed.]

Method 2

$$\text{Let } u = \frac{1}{x} , \text{ so that } x = \frac{1}{u}$$

$$\text{Then } 2\left(\frac{1}{u}\right)^2 + \frac{5}{u} - 9 = 0$$

$$\text{and } 2 + 5u - 9u^2 = 0 \text{ or } 9u^2 - 5u - 2 = 0$$

(2) If the roots of the equation $x^2 + x - 13 = 0$ are α & β , find the equation with roots $2\alpha + 3\beta$ & $3\alpha + 2\beta$

Solution

Let the new equation be $x^2 + bx + c = 0$

$$\text{Then } -b = (2\alpha + 3\beta + 3\alpha + 2\beta) = 5(\alpha + \beta) = 5(-1)$$

$$\text{And } c = (2\alpha + 3\beta)(3\alpha + 2\beta) = 6(\alpha^2 + \beta^2) + 13\alpha\beta$$

$$= 6\{(\alpha + \beta)^2 - 2\alpha\beta\} + 13\alpha\beta = 6(\alpha + \beta)^2 + \alpha\beta$$

$$= 6(-1)^2 - 13 = -7$$

Hence the new equation is $x^2 + 5x - 7 = 0$

(3) If the roots of the equation $x^3 - 14x^2 + 56x - 64 = 0$ are α , β & γ , find the equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ & $\frac{1}{\gamma}$

Solution

Substitution method: Let $u = \frac{1}{x}$, so that $x = \frac{1}{u}$

$$\text{Then } \left(\frac{1}{u}\right)^3 - 14\left(\frac{1}{u}\right)^2 + 56\left(\frac{1}{u}\right) - 64 = 0$$

$$\text{and } 1 - 14u + 56u^2 - 64u^3 = 0$$

$$\text{or } 64u^3 - 56u^2 + 14u - 1 = 0 \text{ (coefficients are reversed)}$$

(4) Find the roots of the equation $x^3 - 14x^2 + 56x - 64 = 0$, given that they form a geometric progression.

Solution

Let the roots be $\frac{\alpha}{r}$, α & $r\alpha$

Then $\frac{\alpha}{r} \cdot \alpha \cdot r\alpha = 64$, so that $\alpha = 4$

Also $\frac{\alpha}{r} + \alpha + r\alpha = 14$, so that $\frac{1}{r} + 1 + r = \frac{7}{2}$

Then $2(1 + r + r^2) = 7r$ and $2r^2 - 5r + 2 = 0$

Hence $(2r - 1)(r - 2) = 0$ and so $r = \frac{1}{2}$ or 2

Thus the roots are 2, 4 and 8.

(5) If the roots of the equation $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ are 5 consecutive positive integers, find expressions for these roots.

Solution

Let the roots be $\alpha - 2$, $\alpha - 1$, α , $\alpha + 1$ & $\alpha + 2$

Then, summing these, $5\alpha = -b$

and hence the roots are $-\left(\frac{b}{5} + 2\right)$, $-\left(\frac{b}{5} + 1\right)$, $-\frac{b}{5}$, $1 - \frac{b}{5}$ & $2 - \frac{b}{5}$

(6) If α , β and γ are the roots of the equation

$$x^3 - 14x^2 + 56x - 64 = 0,$$

find the equation with roots $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$.

Solution

Let $u = \alpha\beta = \frac{\alpha\beta\gamma}{\gamma} = \frac{64}{\gamma}$

Then $\gamma = \frac{64}{u}$ satisfies the original equation

Similarly for $u = \alpha\gamma$ and $u = \beta\gamma$.

Thus the required equation is

$$\left(\frac{64}{u}\right)^3 - 14\left(\frac{64}{u}\right)^2 + 56\left(\frac{64}{u}\right) - 64 = 0,$$

$$\text{giving } 4096 - 896u + 56u^2 - u^3 = 0$$

$$\text{or } u^3 - 56u^2 + 896u - 4096 = 0$$

(7) If α , β and γ are the roots of the equation

$$x^3 - 2x^2 - 4x + 5 = 0,$$

find the equation with roots $\alpha + \beta\gamma$, $\beta + \alpha\gamma$ and $\gamma + \alpha\beta$.

Solution

Let the new equation be $x^3 + bx^2 + cx + d = 0$

$$\text{Then } b = -(\alpha + \beta\gamma + \beta + \alpha\gamma + \gamma + \alpha\beta)$$

$$= -\sum \alpha - \sum \alpha\beta = -2 - (-4) = 2$$

$$\begin{aligned} c &= (\alpha + \beta\gamma)(\beta + \alpha\gamma) + (\alpha + \beta\gamma)(\gamma + \alpha\beta) + (\beta + \alpha\gamma)(\gamma + \alpha\beta) \\ &= (\alpha\beta + \alpha^2\gamma + \beta^2\gamma + \alpha\beta\gamma^2) + \dots \end{aligned}$$

[By symmetry, this contains all the types of terms appearing in the full expansion, and there are $3(4) = 12$ terms.]

$$= \sum \alpha\beta + \sum \alpha^2\beta + \sum \alpha\beta\gamma^2$$

[As a check, this contains $3 + 6 + 3 = 12$ terms]

Thus $c = (-4) + \sum \alpha^2 \beta + \alpha \beta \gamma \sum \alpha$

$$(-4) + \sum \alpha^2 \beta + (-5)(2) = -14 + \sum \alpha^2 \beta \quad (\text{A})$$

[$\sum \alpha^2 \beta$ to be found shortly]

And $d = -(\alpha + \beta \gamma)(\beta + \alpha \gamma)(\gamma + \alpha \beta)$

[this will give $2^3 = 8$ terms]

$$= -(\alpha \beta \gamma + (\sum \alpha^2 \beta^2) + \alpha^2 \beta^2 \gamma^2 + \sum \alpha^3 \beta \gamma)$$

[This can be obtained by performing the expansion, but only noting the types of term (some of which are repeated).]

[$1 + 3 + 1 + 3 = 8$ terms]

$$\text{Thus } d = -(-5) - \sum \alpha^2 \beta^2 - (-5)^2 - \alpha \beta \gamma \sum \alpha^2$$

$$= -20 - \sum \alpha^2 \beta^2 - (-5) \sum \alpha^2 \quad (\text{B})$$

So we need to find $\sum \alpha^2$, $\sum \alpha^2 \beta^2$ & $\sum \alpha^2 \beta$

First of all, consider $(\alpha + \beta + \gamma)^2 = \sum \alpha^2 + 2 \sum \alpha \beta$,

$$\text{so that } \sum \alpha^2 = 2^2 - 2(-4) = 12$$

We can also consider $(\alpha \beta + \alpha \gamma + \beta \gamma)^2 = \sum (\alpha \beta)^2 + 2 \sum \alpha^2 \beta \gamma$

[giving $3 + 2(3) = 9$ terms]

$$\text{so that } \sum \alpha^2 \beta^2 = (-4)^2 - 2 \alpha \beta \gamma \sum \alpha = 16 - 2(-5)(2) = 36$$

Then $(\alpha + \beta + \gamma)(\alpha \beta + \alpha \gamma + \beta \gamma) = (\sum \alpha^2 \beta) + 3 \alpha \beta \gamma$

[$3(2) + 3 = 9$ terms]

so that $\sum \alpha^2 \beta = 2(-4) - 3(-5) = 7$

Hence, from (A), $c = -14 + \sum \alpha^2 \beta = -14 + 7 = -7$

and, from (B),

$$d = -20 - \sum \alpha^2 \beta^2 - (-5) \sum \alpha^2 = -20 - 36 + 5(12) = 4$$

And so the required equation is $x^3 + 2x^2 - 7x + 4 = 0$

[In this example we can use the Factor theorem to see that

α (say) = 1, and that $\beta, \gamma = \frac{1 \pm \sqrt{21}}{2}$, which leads to $\alpha + \beta\gamma$ etc

being $-4, 1$ & 1 , enabling the new equation to be confirmed. In general of course, we may not be able to find a root by the Factor theorem.]