Roots of Polynomial Equations (6 pages; 15/2/17)

(1) Quadratic equations

If $ax^2 + bx + c$ can be factorised as $a(x-\alpha)(x-\beta)$,

or if $ax^2 + bx + c = 0$ has roots α and β

then $ax^2 + bx + c = ax^2 - a(\alpha+\beta)x + a\alpha\beta$

$\Rightarrow \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ (equating coefficients)

Note: This is true for complex roots as well, and also where the coefficients of the equation are complex.

Example 1

If the equation $2x^2 + 5x - 9 = 0$ has roots α and β, find the quadratic equation which has roots $\alpha + 2$ and $\beta + 2$.

Method 1

We know that $\alpha + \beta = -\frac{5}{2}$ and $\alpha\beta = -\frac{9}{2}$

Suppose that the equation we are looking for is $x^2 + bx + c = 0$

Then $(\alpha + 2) + (\beta + 2) = -b$ and $(\alpha + 2)(\beta + 2) = c$

Hence $b = \frac{5}{2} - 4 = -\frac{3}{2}$ and

$c = \alpha\beta + 2\alpha + 2\beta + 4 = -\frac{9}{2} + 2\left(-\frac{5}{2}\right) + 4 = -\frac{11}{2}$

and so the required equation is $x^2 - \frac{3x}{2} - \frac{11}{2} = 0$

or $2x^2 - 3x - 11 = 0$

Method 2 (substitution)

Let $u = x + 2$

Then $x = u - 2$ and

$2x^2 + 5x - 9 = 0 \Leftrightarrow 2(u-2)^2 + 5(u-2) - 9 = 0$

LHS equation $\Leftrightarrow x = \alpha$ or $\beta \Leftrightarrow u = \alpha + 2$ or $\beta + 2$

So the RHS equation $\Leftrightarrow u = \alpha + 2$ or $\beta + 2$

(because the two equations are equivalent).

The RHS equation can be simplified to $2u^2 - 3u - 11 = 0$

Example 2

If the equation $2x^2 + 5x - 9 = 0$ has roots α and β, find the quadratic equation which has roots $\alpha^2 + 1$ and $\beta^2 + 1$, by two methods.

Method 1

$\alpha + \beta = -\frac{5}{2}$ and $\alpha\beta = -\frac{9}{2}$

Let the new equation be $x^2 + bx + c = 0$

Then $\alpha^2 + 1 + \beta^2 + 1 = -b$ and $(\alpha^2 + 1)(\beta^2 + 1) = c$

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{25}{4} + 9$

So $b = -(\frac{25}{4} + 11)$

and $c = (\alpha\beta)^2 + (\alpha^2 + \beta^2) + 1 = \frac{81}{4} + \frac{25}{4} + 10$

Thus the new equation is $4x^2 - 69x + 146 = 0$

Alternative method of finding $\alpha^2 + \beta^2$:

As $2\alpha^2 + 5\alpha - 9 = 0$ and $2\beta^2 + 5\beta - 9 = 0$,

$2(\alpha^2+\beta^2)+5(\alpha+\beta)-18=0,$

so that $2(\alpha^2+\beta^2)+5\left(-\frac{5}{2}\right)-18=0$

and $\alpha^2+\beta^2=\frac{1}{2}(18+\frac{25}{2})$ etc

Method 2

Let $u=x^2+1$, so that $x=\pm\sqrt{u-1}$

Then $2(u-1)\pm5\sqrt{u-1}-9=0$

so that $\pm5\sqrt{u-1}=11-2u$

and $25(u-1)=121-44u+4u^2$

and hence $4u^2-69u+146=0$

Useful result

$(\alpha+\beta)^3=\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$

$=\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$

so that $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

(2) Cubic Equations

Similarly to the quadratic case, if ax^3+bx^2+cx+d can be factorised as $a(x-\alpha)(x-\beta)(x-\gamma)$,

or if $ax^3+bx^2+cx+d=0$ has roots α,β and γ

then $\alpha+\beta+\gamma=-\frac{b}{a}$, $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$ & $\alpha\beta\gamma=-\frac{d}{a}$

Example

If the roots of the equation $x^3-2x^2+5x+3=0$ are

α, β & γ, find the equation with roots α^2,β^2& γ^2

Solution

$\alpha + \beta + \gamma = 2$

$\alpha\beta + \alpha\gamma + \beta\gamma = 5$

$\alpha\beta\gamma = -3$

and $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

We need to find $\alpha^2 + \beta^2 + \gamma^2$, $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2$ and $\alpha^2\beta^2\gamma^2$

$\alpha^2 + \beta^2 + \gamma^2 = 2^2 - 2(5) = -6$ and $\alpha^2\beta^2\gamma^2 = (-3)^2 = 9$

Consider $(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)$

$= \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

So $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = 5^2 - 2(-3)(2) = 37$

Hence the equation is $x^3 + 6x^2 + 37x - 9 = 0$

[Note that the substitution method doesn't work here.]

(3) Higher order polynomials

If $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α, β, γ and δ

then $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$ and $\alpha\beta\gamma\delta = \frac{e}{a}$

These results are often written as

$\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\sum \alpha\beta\gamma = -\frac{d}{a}$, $\alpha\beta\gamma\delta = \frac{e}{a}$

(4) Complex Roots

(i) A cubic curve with real coefficients must intersect the x-axis either once or 3 times (if repeated roots are counted as 2 roots), so that there will be either one or 3 real roots.

Example

Find the roots of the equation $x^3 - x^2 + 8x + 10 = 0$

Solution

Let $f(x) = x^3 - x^2 + 8x + 10$

$f(1) = 18$, $f(-1) = 0$; so $x = -1$ is one root

Then $x^3 - x^2 + 8x + 10 = (x + 1)(x^2 + kx + 10)$

Equating coefficients of x^2: $-1 = 1 + k$, so that $k = -2$

The remaining roots are $\frac{2\pm\sqrt{4-40}}{2} = 1 \pm 3i$

(ii) Conjugate roots

If the coefficients of a polynomial are real, then any complex roots will come in conjugate pairs.

Consider $f(x) = ax^3 + bx^2 + cx + d = 0$

If $z = x + yi$ is a root,

so that $f(z) = a(x + yi)^3 + b(x + yi)^2 + (cx + yi) + d = 0 + 0i$

Then $a(x - yi)^3 + b(x - yi)^2 + (cx - yi) + d$

$= Re(f(z)) - Im(f(z))i = 0 - 0i$

[as i is being replaced with $-i$]

So the complex conjugate is also a root.

Example

If $2+i$ is a root of the equation $x^3 - 7x^2 + 17x - 15 = 0$,

find the remaining roots.

Solution

$2-i$ is also a root

Then, if α is the other root,

$(2+i)+(2-i)+\alpha = 7$, so that $\alpha = 3$