Reference (Pure) (7 pages; 13/1/23)

(A) Numbers

$\sqrt{2} = 1.41421356237309504880168872\ldots$

$e = 2.718281828459045235360287 4\ldots$

Golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618$ (4sf)

(B) Series & expansions

(1) Summations

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$; $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

(2)(i) $x^2 - y^2 = (x-y)(x+y)$

(ii) $x^3 - y^3 = (x-y)(x^2+xy+y^2)$

$x^3 + y^3 = (x+y)(x^2-xy+y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

(iii) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x+y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$, if n is even

$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd

Summary

Factor:	$x-y$	$x+y$
x^n-y^n; odd n	Yes (A)	No **(B)**
x^n-y^n; even n	Yes (C)	Yes (D)
x^n+y^n; odd n	No (P)	Yes **(Q)**
x^n+y^n; even n	No (R)	No (S)

[The 'exceptions' are highlighted. As an aid to memory, the familiar factorisations $x^2-y^2=(x-y)(x+y)$ and

$x^3-y^3=(x-y)(x^2+xy+y^2)$ are examples of (C), (D) & (A), suggesting that $x^3-y^3=(x+y)\ldots$ (ie (B)) is the one that isn't possible for x^n-y^n. This then prompts us to recall that

$x^3+y^3=(x+y)\ldots$ (ie (Q)) is the one that is possible for

x^n+y^n.]

(3) (i) $(a+b+c)^3=(a^3+b^3+c^3)$

$+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)$

$+6abc$

(ii) $(a+b+c)^4=(a^4+b^4+c^4)$

$+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

(iii) $(a+b+c)^n=\sum_{\substack{i,j,k \\ (i+j+k=n)}}\binom{n}{i,j,k}a^ib^jc^k$,

where $\binom{n}{i,j,k}=\frac{n!}{i!j!k!}$

(4) Taylor expansions

(i) Maclaurin: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

(ii) Taylor I: $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$

(iii) Taylor II: $f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \cdots$

[$x = 0$ gives the Maclaurin expansion]

(C) Trigonometry

(1) Powers of Sines and Cosines

[To derive $sin^n\theta$ from $cos^n\theta$, write $sin^n\theta = cos^n(90-\theta)$ etc]

$cos^3\theta = \frac{1}{4}(cos3\theta + 3cos\theta)$

$sin^3\theta = \frac{1}{4}(-sin3\theta + 3sin\theta)$

$cos^4\theta = \frac{1}{8}(cos4\theta + 4cos2\theta + 3)$

$sin^4\theta = \frac{1}{8}(cos4\theta - 4cos2\theta + 3)$

$cos^5\theta = \frac{1}{16}(cos5\theta + 5cos3\theta + 10cos\theta)$

$sin^5\theta = \frac{1}{16}(sin5\theta - 5sin3\theta + 10sin\theta)$

$cos^6\theta = \frac{1}{32}(cos6\theta + 6cos4\theta + 15cos2\theta + 10)$

$sin^6\theta = \frac{1}{32}(-cos6\theta + 6cos4\theta - 15cos2\theta + 10)$

$cos^7\theta = \frac{1}{64}(cos7\theta + 7cos5\theta + 21cos3\theta + 35cos\theta)$

$sin^7\theta = \frac{1}{64}(-sin7\theta + 7sin5\theta - 21sin3\theta + 35sin\theta)$

(2) $cos(n\theta), sin(n\theta)$

[sin($2m\theta$) can't be expressed in terms of powers of $sin\theta$]

$cos3\theta = 4cos^3\theta - 3cos\theta$

$sin3\theta = -4sin^3\theta + 3sin\theta$

$cos4\theta = 8cos^4\theta - 8cos^2\theta + 1$

$cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$

$sin5\theta = 16sin^5\theta - 20sin^3\theta + 5sin\theta$

$cos6\theta = 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$

$cos7\theta = 64cos^7\theta - 112cos^5\theta + 56cos^3\theta - 7\cos\theta$

$sin7\theta = -64sin^7\theta + 112sin^5\theta - 56sin^3\theta + 7sin\theta$

(D) Functions

(1) Reflection in the line $x = a$: $f(x) \to f(2a - x)$

(2) For the cubic $f(x) = ax^3 + bx^2 + cx + d$:

(i) There is always one point of inflexion, at $x = -\frac{b}{3a}$

(ii) Cubic curves have rotational symmetry (of order 2) about the PoI.

(iii) The (x-coordinate of the) PoI lies midway between any turning points.

(iv) The (x-coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(v) There will be two turning points when $b^2 > 3ac$

(E) Geometry & Solids

(1) Tangents and normals to conics

(i) Parabola $y^2 = 4ax$ at $(at^2, 2at)$

tangent: $y = \frac{1}{t}x + at$

normal: $y = -tx + 2at + at^3$

(ii) Rectangular hyperbola $xy = c^2$

tangent: $y = -\frac{1}{t^2}x + \frac{2c}{t}$

normal: $y = t^2x + \frac{c}{t} - ct^3$

(2) Areas & Volumes

(i) Area of sector: $\frac{1}{2}r^2\theta$

(consider limit of area of triangle $\frac{1}{2}r^2 sin\theta$ as $\theta \to 0$)

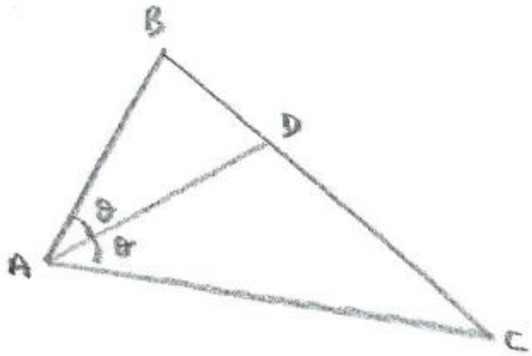
(ii) Volume of sphere: $\frac{4}{3}\pi r^3$

(iii) Volume of pyramid or cone: $\frac{1}{3} \times$ base area $\times$ height

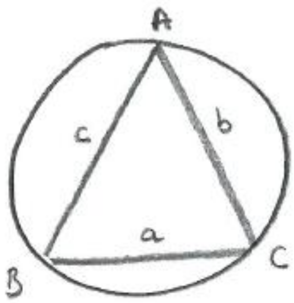
(3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$\frac{BD}{DC} = \frac{AB}{AC}$



(4) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.



As an extension of the Sine rule, $\frac{a}{sinA} = 2R$

(5) Heron's formula for the area of a triangle with sides a, b & c:

$\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$

(F) Hyperbolic Functions

$cosh^2x + sinh^2x = cosh2x$; $cosh^2x - sinh^2x = 1$

$arsinhx = \ln\left(x + \sqrt{x^2+1}\right)$; $arcoshx = \ln\left(x + \sqrt{x^2-1}\right)$ $(x \geq 1)$

$artanhx = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ $(|x| < 1)$

(G) Calculus

(1) Derivatives

$\frac{d}{dx}(a^x) = lna.\,a^x$

(H) Numerical Methods

(1) Simpson's rule

$\int_a^b y\, dx \approx$

$\frac{h}{3}\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}$

where $h = \frac{b-a}{n}$ (n even)