Question Technique (7 pages; 12/9/17)

[for STEP/MAT/TMUA/AEA]

(A) General Tips

(i) Try things that look useful and are quick to do (ie you can quickly establish whether they are leading anywhere).

(ii) Don't do anything that is too obscure: the correct approach, once found, is usually relatively 'simple'.

(iii) Having thought of a method, consider whether there is a quicker alternative, or whether the method can be improved on.

(iv) Re-read the question at critical points:

(a) When about to embark on a solution.

(b) If the solution is not going well.

(c) When you think you've finished the solution (in case there is a supplementary task).

(B) Approaches

(1) Creating equations

(i) Equations can be created from:

(a) information in the question

(b) relevant definitions and theorems

If necessary, create your own variables (for example, a particular length in a diagram).

(ii) In general, we want one equation for each unknown. However, in vector questions, each equation will represent 2 or 3 components (similarly for complex numbers), so that it may be worthwhile creating a new unknown if it means generating 2 or 3 equations.

eg if the vectors $\underline{u}$ & $\underline{v}$ are parallel, we can write $\underline{u} = k\,\underline{v}$

Also, if two polynomials $f(x)$ & $g(x)$ are equivalent (ie equal for all values of x), then we can equate coefficients of powers of x to generate multiple equations.

Note though that a question may have been designed so that one variable cancels out, in which case a smaller number of equations may be acceptable. Also, if only a ratio of two variables is required, one less equation is needed.

(2) Case by case

Example 1: Case 1: $x > 0$ etc

This is a standard way of dealing with the modulus function.

Example 2: Case 1: n is even etc

(3) Reformulating the problem

(i) Example: To sketch the cubic $y = x^3 + 2x^2 + x + 3$, rewrite as $y = x(x^2 + 2x + 1) + 3$

(ii) Example: In order to solve the equation $f(x) = k$, consider where the graph of $y = f(x)$ crosses the line $y = k$

Or, more generally, rearrange an equation to $f(x) = g(x)$, and find where the curves $y = f(x)$ & $y = g(x)$ meet (or show that they won't meet).

(iii) Example: To find the minimum or maximum value of

$y = \frac{f(x)}{g(x)}$, consider what values of k give repeated roots of

$\frac{f(x)}{g(x)} = k$, if it produces a quadratic equation.

(iv) Conversion to a quadratic equation;

eg for $9^x - 3^{x+1} + 2 = 0$, let $y = 3^x$

(4) Experimenting

(i) Try out particular values.

(ii) Consider a concrete example.

(iii) Consider a simpler version of the problem

Example 1:Experiment with a simple function such as $y = x^2$

Example 2: In order to sketch $y = 2^{-x^2}$, start with $y = 2^{-x}$

(iv) Draw a diagram

Sometimes a special feature of a problem may not be revealed until a diagram has been drawn.

(v) Consider extreme cases

Example: Behaviour of function as $x \to \infty$

(5) Using a previous part of the question [relevant to STEP/AEA]

(i) Using a result established in the previous part; possibly after some rearrangement or substitution.

(ii) Applying the same method (possibly modified).

(6) Using information given in the question

[more relevant to STEP/AEA than MAT/TMUA]

(i) Information mentioned explicitly

Example 1: "... where $c \neq 0$": division by c may be involved

Example 2: A condition in the form of an inequality may suggest the use of $b^2 - 4ac$ (especially if it involves a squared quantity).

(ii) Observation of material in the question

(a) to get on the wavelength of the question

(b) for any ideas

Example 1: Part (i) involves 2^{2x-x^2}, and part (iii) involves $2^{-(x-c)^2}$, suggesting that completing the square may help in (i).

Example 2: The presence of a $\pm$ symbol suggests the involvement of a square root.

(C) Devices

(1) Penultimate result

For example, to show that a function $f(n)$ of an integer n cannot be a perfect square, show instead that $f(n) - 1$ is always a perfect square.

(2) Use $2n$ and $2n + 1$ (or $2n - 1$) to represent even and odd numbers respectively (where n is an integer).

Also, use k^2 to represent a non-negative number (which needn't be an integer).

(3) Equating Coefficients

If the polynomials $f(x) = ax^3 + bx^2 + cx + d$ and

$g(x) = ex^3 + fx^2 + gx + h$ are equivalent (ie equal for all values of x), then we can equate coefficients of powers of x, so that

$a = e, b = f, c = g\ \ \& d = h$

(4) Counting

(i) To find the number of ways of doing something (eg arranging digits subject to some constraint ; say the sum of the digits has to be 10): find a systematic way of listing the possibilities, and then of counting the items in the list.

(ii) Note that a method that may be convenient for part (i) of the question may or may not be easily extended to a more complicated part (ii).

(5) Transitional (or 'critical') points

This involves considering the point(s) at which the nature of a problem changes.

Example 1: $\frac{(x-1)(x+2)(x-3)}{(x+1)(x-2)(x+3)} < 0$

The only points at which the sign of the left-hand side can change are at the roots of $(x-1)(x+2)(x-3) = 0$, and at the vertical asymptotes $x = -1, x = 2$ and $x = -3$

Example 2: If investigating a situation involving the intersection of a curve and a straight line, consider first the case where the line touches the curve (ie is a tangent).

(6) Different Classes

To show that two numbers cannot be equal, show that they belong to different classes; eg even and odd numbers.

(7) Substitutions

Integrals and differential equations are commonly simplified by the use of an appropriate substitution. Usually there will be some clue (often in the previous part of the question).

(8) 0, 1 or 2 solutions

If a unique solution is required, or if there is to be exactly 2 solutions, or no solutions, then this suggests the solving of a quadratic equation and considering the discriminant

$b^2 - 4ac$. This is a common theme in the MAT paper.

Example: A tangent $y = mx + c$ to the curve $y = f(x)$ occurs where there is a repeated root of $f(x) = mx + c$

(9) As an alternative to proving that $A \Rightarrow B\ and\ B \Rightarrow A$, it may be easier to prove that $A \Rightarrow B\ and\ A' \Rightarrow B'$ (as $A' \Rightarrow B'$ is equivalent to $B \Rightarrow A$

(10) Simplify a problem by citing WLOG ("without loss of generality").

(11) If an expression can be arranged into the form $(a \pm b)^2$, then this will be non-negative.

(D) Pitfalls

(i) Beware of using ⇒ when ⇔ is required.

(ii) Consider the range of values for which a result is valid (eg if a binomial expansion is involved).

(iii) Beware of losing a solution of an equation by dividing out a factor.

(iv) Beware of spurious solutions (See STEP Problems/D/2).

(v) Beware of multiplying inequalities by a quantity that is (or could be) negative (eg log(0.5)).