Quadratics (4 pages; 5/6/23)

Contents

(A) Quadratic curves

Example: $y = x^2 - 2x - 3$

$x^2 - 2x - 3 = (x + 1)(x - 3)$

Also $x^2 - 2x - 3 = (x - 1)^2 - 4$

The minimum point of $(1, -4)$ lies on the line of symmetry of the curve, which is equidistant from the two roots of $x^2 - 2x - 3 = 0$: -1 & 3.

Also, from the quadratic formula (which is itself derived by completing the square on $ax^2 + bx + c$):

$x = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$

Thus the roots of $x^2 - 2x - 3 = 0$ lie the same distance either side of the line of symmetry of the curve.

(B) Factorising $ax^2 + bx + c$

(i) If unsure whether the expression will factorise at all, examine

$b^2 - 4ac$. If it is not a perfect square, then no factorisation involving integers will be possible.

(ii) If a and/or c are prime numbers, then the factorisation can often be carried out 'by inspection'; ie there will only be a few possibilities to try out.

Example : $5x^2 - 34x - 7$

First of all, $b^2 - 4ac = 1156 + 4(5)(7) = 1296 = 36^2$, and an integer factorisation is possible.

The factorisation has to be of the form:

$(5x + p)(x + q)$

where either $p = 7, q = -1 \quad or \quad p = -1, q = 7$

or $p = 1, q = -7 \quad or \quad p = -7, q = 1$

Clearly, only $q = -7$ is likely to give rise to the $-34x$

(considering the term $5xq$), so the factorisation must be

$(5x + 1)(x - 7)$

(iii) Where the factorisation is less easy to arrive at, and as an alternative to using the quadratic formula, the following procedure can be applied:

In general, let $ax^2 + bx + c = ax^2 + Dx + Ex + c$

such that $D + E = b$ and $DE = ac$

This is the extension of the familiar method when $a = 1$.

If an integer factorisation exists, then suitable D & E can be found, and $ax^2 + Dx$ and $Ex + c$ will always share a common factor. See below for a proof.

Example : $f(x) = 6x^2 + x - 12$

We need to find D & E such that $D + E = 1$ and $DE = -72$

$D = 9$ & $E = -8$ satisfy this

Then $f(x) = 6x^2 + 9x - 8x - 12$

$= 3x(2x + 3) - 4(2x + 3)$

$= (3x - 4)(2x + 3)$

Alternatively, $f(x) = 6x^2 - 8x + 9x - 12$

$= 2x(3x - 4) + 3(3x - 4)$

$= (2x + 3)(3x - 4)$

Note: In more awkward cases, it is possible to home in on the correct values of D & E by first of all limiting our choices to those where (in this case) $DE = -72$, and at each stage observing how far out $D + E$ is, and whether the factors of -72 need to be closer together or further apart.

Proof

Suppose that we can find D & E such that $D + E = b$ and

$DE = ac$

We are trying to factorise $ax^2 + Dx$ and $Ex + c$

So let $a = pm$ and $D = pn$, where p is the HCF of a & D

Similarly let $E = qu$ and $c = qv$, where q is the HCF of E & c.

Then $ax^2 + Dx + Ex + c$ can be written as

$px(mx+n)+q(ux+v)$

and we need to show that $m=u$ & $n=v$.

Now $DE=ac$, so that $pnqu=pmqv$, and hence $nu=mv$

(provided that p & q are non-zero, which is the case as long as

a & c are non-zero).

Now, m & n have no factors in common (as p is the HCF of a & D). Hence the factors of n must be contained within v (since

$nu=mv$), and so $n\leq v$.

Also, u & v have no factors in common (as q is the HCF of E & c). Hence the factors of v must be contained within n (since

$nu=mv$), and so $v\leq n$.

As $n\leq v$ and $v\leq n$, it follows that $n=v$, and hence $m=u$

(since $nu=mv$), as required.