Pure - Miscellaneous: Exercises (Sol'ns)(7 pages; 17/1/20)

Contents

(A) Indices

(1*) (i) Does $\sqrt{4}$ equal 2 or ± 2? (ii) Simplify $\sqrt{x^2}$

Solution

(i) By convention, 2 (consider the $\pm$ in the quadratic formula).

(ii) $|x|$

(2*) Simplify $\left(1+\left(1+2^{-\frac{1}{2}}\right)^{-1}\right)^{-1}$

Solution

$\left(1+\left(1+2^{-\frac{1}{2}}\right)^{-1}\right)^{-1} = \frac{1}{1+\frac{1}{1+\frac{1}{\sqrt{2}}}} = \frac{1}{1+\frac{\sqrt{2}}{\sqrt{2}+1}} = \frac{\sqrt{2}+1}{\sqrt{2}+1+\sqrt{2}} = \frac{\sqrt{2}+1}{2\sqrt{2}+1} =$

$\frac{(\sqrt{2}+1)(2\sqrt{2}-1)}{(2\sqrt{2}+1)(2\sqrt{2}-1)}$

$= \frac{4-\sqrt{2}+2\sqrt{2}-1}{8-1} = \frac{3+\sqrt{2}}{7}$

(B) Partial Fractions

(1***) Express $\frac{1}{(1-x^2)^2}$ in terms of partial fractions

Solution

$\frac{1}{(1-x^2)^2} = \frac{1}{(1-x)^2(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2}$

so that $1 = A(1-x)(1+x)^2 + B(1+x)^2 + C(1+x)(1-x)^2 + D(1-x)^2$

Then $x = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

$x = -1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$

$x = 0 \Rightarrow 1 = A + B + C + D \Rightarrow A + C = \frac{1}{2}$

Equating coefficients of $x^3 \Rightarrow 0 = -A + C$

Hence $A = C = \frac{1}{4}$

and $\frac{1}{(1-x^2)^2} = \frac{1}{4(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{4(1+x)} + \frac{1}{4(1+x)^2}$

(C) Recurrence relations

(1#) Consider the sequence defined by $u_n = au_{n-1} + b$,

where a & b are real constants, and u_0 is given.

(i) What familiar sequences are special cases of this sequence?

Solution

Setting $a = 1$ gives an arithmetic sequence.

Setting $b = 0$ gives a geometric sequence.

(ii) Define a new sequence by $v_n = u_n + c$

For what value of c, in terms of a & b, will v_n be a geometric sequence?

For what value of a does this not work?

Solution

$v_{n-1} = u_{n-1} + c$, and hence

$u_n = au_{n-1} + b \Rightarrow v_n - c = a(v_{n-1} - c) + b$

$\Rightarrow v_n = av_{n-1} + b + c(1 - a)$

For v_n to be a geometric sequence, we want $b + c(1 - a) = 0$,

so that $c = \frac{-b}{1-a} = \frac{b}{a-1}$, provided that $a \neq 1$

When $a = 1$, u_n, and hence v_n also, are arithmetic sequences.

(iii) If $u_n = 2u_{n-1} + 3$, and $u_0 = 4$, find a formula for u_n in terms of n

Solution

From (ii), $c = \frac{3}{2-1} = 3$ and $v_n = 2v_{n-1}$

Then $v_n = v_0(2^n)$

and $v_n = u_n + 3$, so that $u_n + 3 = (u_0 + 3)(2^n)$

and $\therefore u_n = 7(2^n) - 3$

(and this can be checked by comparing with $u_n = 2u_{n-1} + 3$, and $u_0 = 4$)

(iv) Find a similar formula for $u_n = au_{n-1} + b$, where u_0 is given.

Solution

From (ii), $c = \frac{b}{a-1}$ and $v_n = av_{n-1}$

Then $v_n = v_0(a^n)$

and $v_n = u_n + c$, so that $u_n + c = (u_0 + c)(a^n)$

and $\therefore u_n = (u_0 + c)(a^n) - c = \left(u_0 + \frac{b}{a-1}\right)(a^n) - \frac{b}{a-1}$

(v) Under what conditions will u_n be constant? Give a non-trivial example.

Solution

Either $a = 1$ & $b = 0$

Or $a = 0$ and $u_0 = b$

Or $u_0 + \frac{b}{a-1} = 0$; ie $u_0 = \frac{b}{1-a}$

For example, $u_n = 2u_{n-1} - 1$, where $u_0 = 1$

(Z) Miscellaneous

(1*) How are the following usually defined?

(a) Whole numbers (b) Natural numbers

Solution

(a) Whole numbers: Integers (including zero and negative integers)

(b) Natural numbers: usually positive integers, but sometimes including zero

(2#) Prove that $E' \Rightarrow L'$ is equivalent to $L \Rightarrow E$

Solution

Suppose that L is true & E is not true; then $E' \Rightarrow L'$ means that L is not true; ie a contradiction; hence $L \Rightarrow E$

(3#) What is a transcendental number?

Solution

First of all, an 'algebraic number' is one that is the root of a polynomial equation with integer coefficients.

Irrational numbers can be divided into two classes: those that are algebraic numbers and those that aren't. The former are called surds and the latter are called 'transcendental numbers'. The best known examples of transcendental numbers are π and e.

(4#) Find the square roots of $49 - 12\sqrt{5}$

Solution

Let $x^2 = 49 - 12\sqrt{5}$

Consider $x = a + b\sqrt{5}$

Then $a^2 + 2ab\sqrt{5} + 5b^2 = 49 - 12\sqrt{5}$

Let $a^2 + 5b^2 = 49$ & $2ab = -12$

[a variation on Equating Coefficients]

Looking for integer solutions, we see that either

$a = 2$ & $b = -3$ or $a = -2$ & $b = 3$ work.

(5#) Show that $\sum_{r=0}^{n}\binom{n}{r} = 2^n$

Solution

Method 1: Consider $(1+1)^n$

Method 2: Pascal's triangle

The sum of each row is twice the sum of the previous one.

eg $1+5+10+10+5+1$

$= (1+10+5)[alternate\ terms] + (5+10+1)$

$= 2(1+10+5) = 2(1+[4+6]+[4+1])$

& $1+6+15+20+15+6+1$

$= (1+15+15+1) + (6+20+6)$

$= (1+[5+10]+[10+5]+1)$

$+([1+5]+[10+10]+[5+1])$

Method 3: Counting ways of selecting any number of items

1st counting method: $\sum_{r=0}^{n}\binom{n}{r}$

2nd counting method: For each object, there are 2 choices: include or exclude; giving 2^n

[Note: 1 way of choosing no objects is included in the total.]

Method 4: Induction

If true for $n = k$, so that $\sum_{r=0}^{k}\binom{k}{r} = 2^k$,

then $\sum_{r=0}^{k+1}\binom{k+1}{r} = \binom{k+1}{0} + \{\sum_{r=1}^{k}\binom{k+1}{r}\} + \binom{k+1}{k+1}$

$= 1 + \sum_{r=1}^{k}\left\{\binom{k}{r-1} + \binom{k}{r}\right\} + 1$

$= 1 + \{\sum_{r-1=0}^{k-1} \binom{k}{r-1}\} + [\{\sum_{r=0}^{k} \binom{k}{r}\} - \binom{k}{0}] + 1$

$= 1 + \{\sum_{R=0}^{k-1} \binom{k}{R}\} + [2^k - 1] + 1$

$= 1 + \{\sum_{R=0}^{k} \binom{k}{R}\} - \binom{k}{k} + 2^k$

$= 1 + 2^k - 1 + 2^k$

$= 2^{k+1}$