Polynomials - Exercises (Sol'ns) (4 pages; 7/10/18)

(1) What is the minimum value of $(x^2 - 4x + 3)(x^2 + 4x + 3)$, where x can be any real number? (without using Calculus)

Solution

$(x^2 - 4x + 3)(x^2 + 4x + 3) = (x - 3)(x - 1)(x + 3)(x + 1)$

$= (x^2 - 9)(x^2 - 1)$

$= (x^2 - 5 - 4)(x^2 - 5 + 4)$

$= (x^2 - 5)^2 - 16$

which has -16 as its minimum value

Alternative approaches

(i) … $(x^2 - 9)(x^2 - 1) = x^4 - 10x^2 + 9$

$= (x^2 - 5)^2 - 16$

(ii) $(x^2 - 4x + 3)(x^2 + 4x + 3)$

$= x^4 + x^3(4 - 4) + x^2(3 - 16 + 3) + x(-12 + 12) + 9$

$= x^4 - 10x^2 + 9$

$= (x^2 - 5)^2 - 16$

(2) (i) Factorise (a) $x^3 - y^3$ (b) $x^3 + y^3$

(ii) Can $3^{54} - 2^{54}$ be prime?

Solution

(i)(a) Let $f(x) = x^3 - y^3$

By the Factor theorem (treating $f(x)$ as a cubic in x), since

$f(y) = 0$, $(x - y)$ is a factor of $x^3 - y^3$, leading to

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(b) Similarly, $(x + y)$ is a factor of $x^3 + y^3$, leading to

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

[More generally,

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

and, if n is odd:

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$]

(ii) We could consider using the result

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

but it isn't of any use having $x - y = 3 - 2 = 1$.

However, we can write $3^{54} - 2^{54}$ as $(3^{18})^3 - (2^{18})^3$, for example, to give the factor $3^{18} - 2^{18}$ (similarly, $3^3 - 2^3$ is also a factor).

[Alternatively, we could just write $3^{54} - 2^{54}$ as $(3^{27})^2 - (2^{27})^2$, and use the difference of two squares.]

So $3^{54} - 2^{54}$ isn't a prime number.

(3) (i) Find an expansion for $(a + b + c)^3$, and give a justification for the coefficients.

(ii) Extend this to $(a+b+c)^4$

Solution

(i) By an ordinary expansion:

$(a+b+c)^3 = ([a+b]+c)^3$

$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$

$= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc)$

$+(3ac^2 + 3bc^2) + c^3$

$= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an a^3 term from

$(a+b+c)(a+b+c)(a+b+c)$; namely by choosing the a from each of the 3 brackets.

There are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]× 1[number of ways of choosing two as from the remaining 2 brackets].

Finally, there are 6 ways of creating an abc term: 3[number of ways of choosing the a]× 2[number of ways of choosing the b from the remaining 2 brackets]× 1[number of ways of choosing the c from the remaining bracket].

The final expression then follows by symmetry.

(ii) $(a+b+c)^4 = (a^4 + b^4 + c^4)$

$+4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$

$+6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)$

For the a^2b^2 term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets from $(a + b + c)(a + b + c)(a + b + c)(a + b + c)$

to give a^2, and then just 1 way of obtaining the b^2 term.

For the a^2bc term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets for the a^2 again, multiplied by the 2 ways of choosing brackets for the b and c.

For further investigation: the 'trinomial' expansion of $(a + b + c)^n$ can be shown to be $\sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1 + a_2 + \cdots + a_m)^n$)

(4) What can be said about the graph of $f(x)$ if $(x - a)^n$ is a factor of $f(x)$, where $f(x)$ is a polynomial function and $n \in \mathbb{Z}^+$?

Solution

There is a turning point at $x = a$ if n is even, and a point of inflexion if $n > 1$ is odd.