Polar Curves – Q4 [Practice/H](12/6/21)

(i) Sketch the curve $r^2 = sin2\theta$.

(ii) Show how to sketch the curve $r^2 = cos2\theta$ by applying a transformation to $r^2 = sin2\theta$.

(iii) Find the largest y-coordinate of the curve $r^2 = sin2\theta$.

Solution

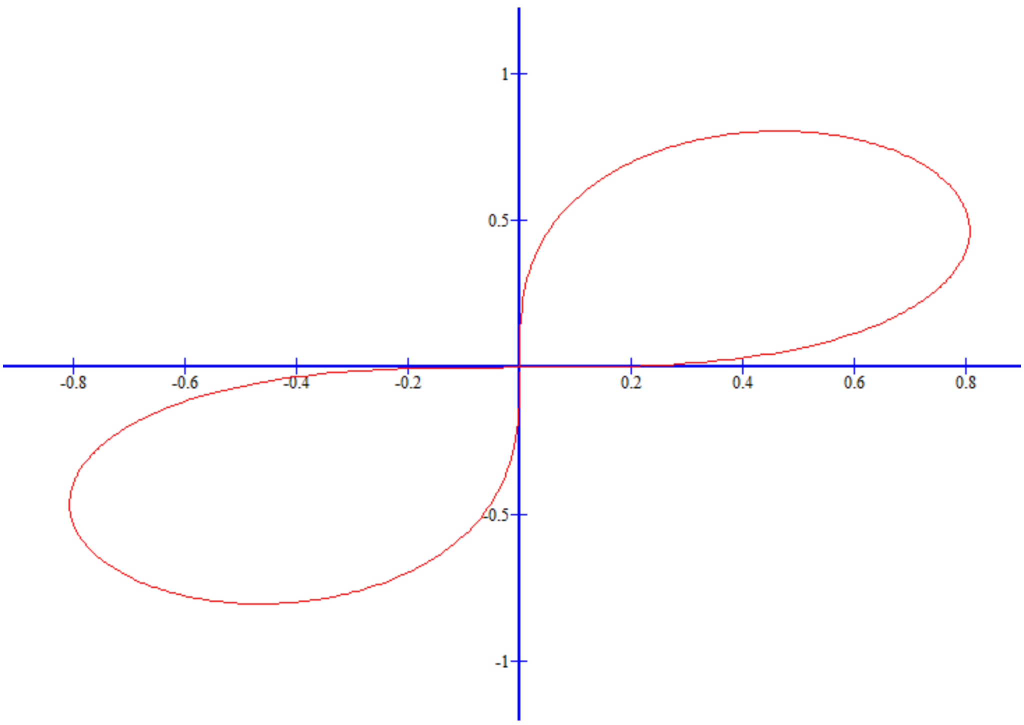
(i) Step 1: As $r = \pm\sqrt{sin2\theta}$ isn't a function of either $cos\theta$ or $sin\theta$, there is no symmetry about the x or y axis.

Step 2: The curve isn't defined for $\frac{\pi}{2} < \theta < \pi$ or for $\frac{3\pi}{2} < \theta < 2\pi$ (as $sin2\theta < 0$).

Step 3: For each θ there will positive and negative values of r of the same magnitude. [However the negative values of r for θ will overlap with the positive values for $\theta + \pi$.]

Step 4: Key points to plot are: $\theta = 0, r = 0$; $\theta = \frac{\pi}{4}, r = \pm 1$; $\theta = \frac{\pi}{2}, r = 0$ (and the cycle repeats itself for $\theta = \pi$ to $\theta = \frac{3\pi}{2}$).

Step 5: The gradient at $\theta = 0$ (when $r = 0$) is 0 (ie along the line $\theta = 0$), and at $\theta = \frac{\pi}{2}$ it is ∞ (ie along the line $\theta = \frac{\pi}{2}$).



(ii) $r = 1$ when $\theta = \frac{\pi}{4}$ for $r^2 = sin2\theta$, and when $\theta = 0$ for $r^2 = cos2\theta$, so the curve for $r^2 = sin2\theta$ needs to be rotated by $\frac{\pi}{4}$ clockwise.

[This rotation transforms $r^2 = sin2\theta$ to $r^2 = sin2(\theta + \frac{\pi}{4})$ [as clockwise is the negative direction] $= \sin\left(2\theta + \frac{\pi}{2}\right) = cos2\theta$]

(iii) $y = rsin\theta, r^2 = sin2\theta \Rightarrow y^2 = sin2\theta. sin^2\theta$

$\Rightarrow 2y\frac{dy}{d\theta} = 2cos2\theta sin^2\theta + sin2\theta(2sin\theta cos\theta)$

Then $\frac{dy}{d\theta} = 0 \Rightarrow (cos^2\theta - sin^2\theta)sin^2\theta + (2sin\theta cos\theta)(sin\theta cos\theta) = 0$

$\Rightarrow 3cos^2\theta sin^2\theta - sin^4\theta = 0$

$\Rightarrow sin^2\theta(3cos^2\theta - sin^2\theta) = 0$

$\Rightarrow sin^2\theta(3cos^2\theta - (1 - cos^2\theta)) = 0$

$\Rightarrow sin^2\theta(4cos^2\theta - 1) = 0$

$\Rightarrow \theta = 0 \; or \; \pi$ (within $[0, 2\pi)$) (ie when the curve is at the Origin)

or $cos\theta = \pm\frac{1}{2}$, so that $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3} \; or \; \frac{5\pi}{3}$

The curve doesn't exist for $\theta = \frac{2\pi}{3} \; and \; \frac{5\pi}{3}$, and so the required value is

$\theta = \frac{\pi}{3}$ (when the y-coordinate is positive).

At $\theta = \frac{\pi}{3}$, $y^2 = sin2\theta. sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^3$