Partial Fractions (10 pages; 8/8/17)

(1) Overview

The topic of 'partial fractions' refers to the rearrangement of expressions of certain types into a sum of simpler ('partial') fractions.

Thus we will see that:

(A) $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ (where p or q could be zero, and a & b could be positive, negative or zero)

(B) $\frac{px^2+qx+r}{(x-a)(x-b)^2} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{(x-b)^2}$

(C) $\frac{px^2+qx+r}{(x-a)(x^2+b^2)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}$ (where B or C might be zero)

(with similar results where there are further factors in the denominator)

Notes

(i) The terms "Type A" etc are not standard, and shouldn't be referred to in exam answers.

(ii) The methods to be described can also be applied where the factors are of the form $ax+b$ or ax^2+b^2

(2) Type A

Example: $\frac{1}{x(x+1)}$

This can be rearranged into the form $\frac{A}{x} + \frac{B}{x+1}$, as follows:

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)}{x(x+1)} + \frac{Bx}{x(x+1)}$$

so that $1 = A(x+1) + B(x)$ (*)

There are 3 general methods for finding A and B:

Method 1

Set x equal to -1 and 0 in (*), in turn.

Thus: $x = -1 \Rightarrow 1 = -B \;\&\; B = -1$

whilst $x = 0 \Rightarrow 1 = A$

So $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

Method 2

Equating coefficients of powers of x in (*):

$x^1: 0 = A + B$

$x^0: 1 = A$ [We are taking the constant terms of each side. This has the same effect as setting x equal to 0.]

Method 3

Setting x equal to any convenient values.

We've already used $x = -1 \;\&\; x = 0$ in Method 1, but another value could be used as a check.

(3) Example: $\frac{2x^2+3x+1}{(x-1)(x+2)(x-3)}$

Although the numerator is more complicated, and we have an additional factor in the denominator, this example is again of Type A, and the same methods can be applied.

Thus $\frac{2x^2+3x+1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$= \frac{A(x+2)(x-3)}{(x-1)(x+2)(x-3)} + \frac{B(x-1)(x-3)}{(x-1)(x+2)(x-3)} + \frac{C(x-1)(x+2)}{(x-1)(x+2)(x-3)}$

so that $2x^2 + 3x + 1$

$= A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

Method 1

$x = -2 \Rightarrow 3 = B(-3)(-5) \Rightarrow B = \frac{1}{5}$

$x = 3 \Rightarrow 28 = C(2)(5) \Rightarrow C = \frac{14}{5}$

$x = 1 \Rightarrow 6 = A(3)(-2) \Rightarrow A = -1$

Method 2

$x^2: 2 = A + B + C$

$x^1: 3 = -A - 4B + C$

$x^0: 1 = -6A + 3B - 2C$

Method 3

Again, another value such as $x = -1$ could be used as a check.

Thus $\frac{2x^2+3x+1}{(x+1)(x+2)(x-3)} = -\frac{1}{x-1} + \frac{B}{5(x+2)} + \frac{14C}{5(x-3)}$

Note that Method 1 is the simplest to apply.

(4) Example: $\frac{x^2+3}{(x-2)(x+1)}$

A problem arises here. If we attempt $\frac{x^2+3}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)}$, then we obtain:

$x^2 + 3 = A(x+1) + B(x-2)$,

and we see that there is no x^2 term on the RHS.

In the case of any fraction where the order (ie the highest power of x) of the numerator is greater than or equal to the order of the denominator, we will need to do an initial rearrangement.

There are two possible approaches:

Approach 1

$$\frac{x^2+3}{(x-2)(x+1)} = \frac{x^2+3}{x^2-x-2} = \frac{x^2-x-2}{x^2-x-2} + \frac{x+5}{x^2-x-2} = 1 + \frac{x+5}{(x-2)(x+1)}$$

and then $\frac{x+5}{(x-2)(x+1)}$ can be broken down into partial fractions as usual

Approach 2

As we can anticipate that the fraction can be rearranged into the form $1 + \frac{px+\mathrm{q}}{(x-2)(x+1)}$, and that $\frac{px+\mathrm{q}}{(x-2)(x+1)}$ can be written as

$\frac{A}{(x-2)} + \frac{B}{(x+1)}$, we can go straight to

$$\frac{x^2+3}{(x-2)(x+1)} = 1 + \frac{A}{(x-2)} + \frac{B}{(x+1)}$$

Then the RHS can be written as $\frac{(x-2)(x+1)+A(x+1)+B(x-2)}{(x-2)(x+1)}$

and equating the two sides gives:

$$x^2 + 3 = (x-2)(x+1) + A(x+1) + B(x-2)$$

Any combination of the standard 3 methods can then be applied.

Example: $\frac{2x^3-3x^2+4}{(x-2)(x+1)}$

Approach 1

$\frac{2x^3-3x^2+4}{(x-2)(x+1)} = \frac{2x^3-3x^2+4}{x^2-x-2}$

$= \frac{2x(x^2-x-2)}{x^2-x-2} + \frac{-x^2+4x+4}{x^2-x-2} = 2x + \frac{-(x^2-x-2)}{x^2-x-2} + \frac{3x+2}{x^2-x-2}$

$= 2x - 1 + \frac{3x+2}{(x-2)(x+1)}$

(and then the usual Type A procedure can be followed)

Approach 2

$\frac{2x^3-3x^2+4}{(x-2)(x+1)} = 2x + A + \frac{B}{x-2} + \frac{C}{x+1}$, so that

$2x^3 - 3x^2 + 4 = (2x + A)(x-2)(x+1) + B(x+1) + C(x-2)$

and then the usual methods can be applied.

(5) Type B

Example $\frac{2x^2+3x-1}{(x-1)(x-2)^2}$

It isn't immediately obvious that this can be rewritten as

$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

However, by applying the standard methods, we can find values for A, B & C, to show that the rearrangment does in fact work.

Thus $2x^2 + 3x - 1 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

However, we note here that, because of the repeated factor, Method 1 (substituting $x = 2$ & $x = 1$) can only be used to create two equations; so that either Method 2 (equating coefficients of powers of x) or Method 3 (substituting another convenient value for x) will have to be used as well.

Method 2 can in fact be used on its own, as it will generate 3 equations (for the 3 unknowns A, B & C). However, Method 1 is usually simplest, and an additional equation is also advisable, as a check.

Thus, Method 1 gives:

$13 = C$ & $4 = A$

whilst equating the constant terms (or substituting $x = 0$) gives:

$-1 = 4A + 2B - C$, so that $-1 = 16 + 2B - 13$

and hence $B = -2$

Then, equating coefficients of x^2 (as a check) gives:

$2 = A + B$ (which agrees with the values obtained),

and (as a further check), equating coefficients of x gives:

$3 = -4A - 3B + C$ (which again agrees with the values obtained).

(6) Type C

Example: $\frac{x^2+1}{x(x^2+2)}$

For Type C, the solution $\frac{A}{x-a} + \frac{B}{x^2+b^2}$ won't generally work.

In this example, we would have $x^2 + 1 = A(x^2 + 2) + Bx$,

forcing A to be 1 (by equating coefficients of x^2). But this means that the constant terms don't match. Thus, we don't have enough flexibility, and another constant C is required, so that the RHS becomes $\frac{A}{x-a} + \frac{Bx+C}{x^2+b^2}$, leading in this example to

$x^2+1=A(x^2+2)+(Bx+C)x \quad (*)$

Setting $x=0$ gives $1=2A$

Equating coefficients of x^2 gives $1=A+B$

Thus, $A=\frac{1}{2}$ & $B=\frac{1}{2}$

Equating coefficients of x: $0=C$

Thus, $\frac{x^2+1}{x(x^2+2)}=\frac{1}{2x}+\frac{x}{2(x^2+2)}$

[and checks can be made by substituting other values for x in (*)]

Notes

(i) A denominator of the form x^2-b^2(rather than x^2+b^2) could be factorised into linear terms.

(ii) A denominator of the form x^2+bx+c could either be factorised (if $b^2-4c\geq 0$; with the factors involving surds if necessary) or rewritten in the form $(x-p)^2+q^2$, by completing the square (and a variation of Type C could then be applied). For this reason, a 3 term quadratic should not be encountered in the denominator.

(7) Expressions such as $\frac{x+2}{x+1}$, $\frac{2-3x}{x+1}$ or $\frac{x^2+2x}{x+1}$ (with a single factor in the denominator) can be rearranged as follows:

$\frac{x+2}{x+1}=1+\frac{1}{x+1}$

$\frac{2-3x}{x+1}=-\frac{3(x+1)}{x+1}+\frac{5}{x+1}=\frac{5}{x+1}-3$

$\frac{x^2+2x}{x+1}=\frac{x(x+1)}{x+1}+\frac{x}{x+1}=x+\frac{x+1}{x+1}+\frac{-1}{x+1}=x+1-\frac{1}{x+1}$

(8) Applications of Partial Fractions

(i) Integration, as expressions of the form $\frac{A}{x-a}$ and $\frac{C}{(x-b)^2}$ can readily be integrated; whilst $\frac{Bx+C}{x^2+b^2} = \frac{Bx}{x^2+b^2} + \frac{C}{x^2+b^2}$

The 1st term can then be integrated via the substitution $u = x^2$, and the 2nd term integrates to $C\left(\frac{1}{b}\right)\arctan(\frac{x}{b})$.

(ii) Binomial expansions, as the individual partial fractions can be expanded binomially.

(9) Exercises (with solutions)

(9.1) Determine $\int \frac{2x^2-x+5}{(x-3)(x+2)^2}\, dx$

Solution

$\frac{2x^2-x+5}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

so that $2x^2 - x + 5 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$

$x = -2 \Rightarrow 15 = -5C; C = -3$

$x = 3 \Rightarrow 20 = 25A; A = \frac{4}{5}$

Equating coefficients of x^0: $5 = 4A - 6B - 3C$

$\Rightarrow B = \frac{1}{6}\left(\frac{16}{5} + 9 - 5\right) = \frac{6}{5}$

[Check: Equating coefficients of x^2: $2 = A + B$

and $x = 1 \Rightarrow 6 = 9A - 6B - 2C$]

Then $\int \frac{2x^2-x+5}{(x-3)(x+2)^2}\, dx = \int \frac{4}{5(x-3)} + \frac{6}{5(x+2)} - \frac{3}{(x+2)^2}\, dx$

$= \frac{4}{5}ln|x-3| + \frac{6}{5}ln|x+2| + \frac{3}{x+2} + c$

(9.2) Determine $\int_0^{\sqrt{2}} \frac{(x+1)(x^2+1)}{(x+2)(x^2+2)} dx$

Solution

As the degree of the numerator is not less than that of the denominator, a rearrangement is necessary, in order for the standard approach to be applied. This is done in Method 1, although Method 2 is quicker.

Method 1

$\frac{(x+1)(x^2+1)}{(x+2)(x^2+2)} = \frac{x^3+x^2+x+1}{x^3+2x^2+2x+4} = \frac{x^3+2x^2+2x+4}{x^3+2x^2+2x+4} - \frac{x^2+x+3}{x^3+2x^2+2x+4}$

$= 1 - \frac{x^2+x+3}{(x+2)(x^2+2)}$

$= 1 - \frac{A}{x+2} - \frac{Bx+C}{x^2+2}$

where $A(x^2+2) + (Bx+C)(x+2) = x^2+x+3$

Then setting $x = -2$ gives $6A = 5$, so that $A = \frac{5}{6}$

Also, equating constant terms: $2A + 2C = 3$, so that $C = \frac{1}{2}\left(3 - \frac{5}{3}\right) = \frac{2}{3}$

and equating coefficients of x^2: $A + B = 1$, so that $B = \frac{1}{6}$

[Check: equating coefficients of x: $2B + C = 1$

and setting $x = 1$ gives $3A + 3B + 3C = 5$]

Method 2

$\frac{(x+1)(x^2+1)}{(x+2)(x^2+2)} = \frac{x^3+x^2+x+1}{(x+2)(x^2+2)} = 1 + \frac{D}{x+2} + \frac{Ex+F}{x^2+2}$

where $(x+2)(x^2+2)+D(x^2+2)+(Ex+F)(x+2)=x^3+x^2+x+1$

Then, setting $x=-2$ gives $6D=-5$, so that $D=-\frac{5}{6}$

Also, equating constant terms: $4+2D+2F=1$, so that $F=\frac{1}{2}\left(-3+\frac{5}{3}\right)=-\frac{2}{3}$

and equating coefficients of x^2: $2+D+E=1$, so that $E=-\frac{1}{6}$

Then $\int_0^{\sqrt{2}} \frac{(x+1)(x^2+1)}{(x+2)(x^2+2)} dx = \int_0^{\sqrt{2}} 1 - \frac{5}{6(x+2)} - \frac{x+4}{6(x^2+2)} dx$

$= [x - \frac{5}{6} ln(x+2) - \frac{1}{12} ln(x^2+2) - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right) arctan\left(\frac{x}{\sqrt{2}}\right)]_0^{\sqrt{2}}$

$= \left(\sqrt{2} - \frac{5}{6} ln\left(2+\sqrt{2}\right) - \frac{1}{12} ln4 - \frac{\sqrt{2}}{3}\left(\frac{\pi}{4}\right)\right) - (-\frac{5}{6} ln2 - \frac{1}{12} ln2)$

$= \sqrt{2} - \frac{5}{6} ln\left(2+\sqrt{2}\right) + \frac{3}{4} ln2 - \frac{\pi\sqrt{2}}{12}$