

Parabolas - Exercises (Solutions) (7 pages; 18/8/19)

(1) Using the parametric equations of a parabola ($x = at^2$, $y = 2at$), show that the midpoints of chords of a parabola that have the same direction lie on a straight line parallel to the x -axis.

[A chord of a parabola joins two points on the parabola.]

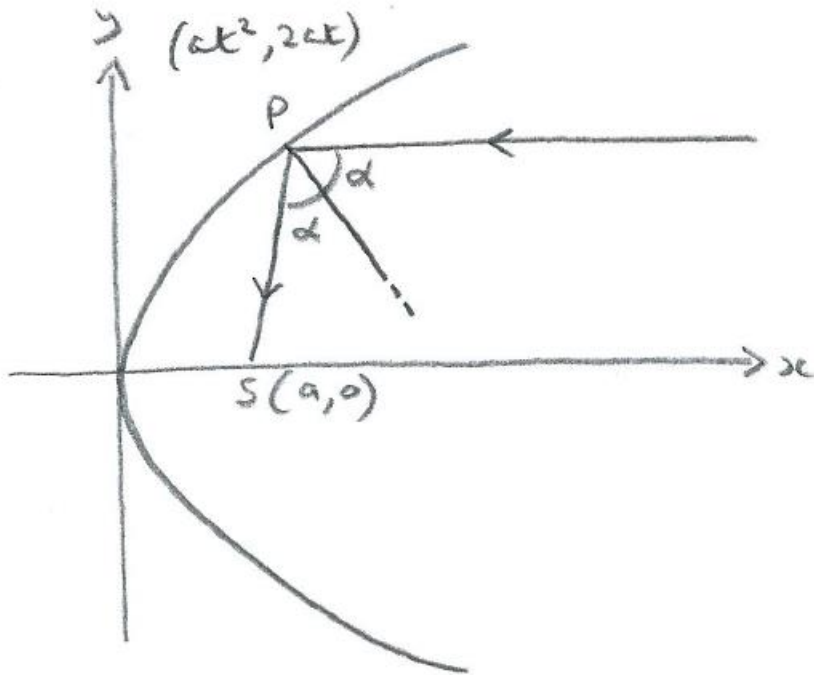
Solution

Let points P and Q on the parabola have parameters t_1 & t_2 .

The chord PQ has gradient $\frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2(t_2 - t_1)}{t_2^2 - t_1^2} = \frac{2}{t_1 + t_2}$, and we are told that this is constant.

The y -coordinate of the midpoint of PQ is $\frac{1}{2}(2at_1 + 2at_2) = a(t_1 + t_2)$, which is constant, as $\frac{2}{t_1 + t_2}$ and therefore $t_1 + t_2$ are constant, giving the required result.

(2) A ray (eg of light) travels on a path parallel to the x -axis and hits the surface of the parabola $y^2 = 4ax$ at the point P ($at^2, 2at$). The angle between the incoming ray and the normal at P is α . It can be assumed that the angle that the reflected ray makes with the normal is also α .



- (i) Show that $\tan \alpha = t$
- (ii) Find the gradient of the reflected ray.
- (iii) Show that the reflected ray passes through the focus of the parabola.

Solution

(i) The gradient of the tangent at P is $\frac{1}{t}$ (standard result), and hence the gradient of the normal is $-t$.

Hence, as the normal makes an angle $\pi - \alpha$ with the positive x -axis, $-t = \tan(\pi - \alpha) = -\tan \alpha$, so that $\tan \alpha = t$.

(ii) The gradient of the reflected ray is $\tan(\pi - 2\alpha) = -\tan(2\alpha)$

$$= \frac{-2\tan \alpha}{1 - \tan^2 \alpha} = \frac{2t}{t^2 - 1}$$

The equation of the reflected ray is $y - 2at = \frac{2t}{t^2 - 1}(x - at^2)$.

When it meets the x -axis, $-2at = \frac{2t}{t^2 - 1}(x - at^2)$,

$$\text{and } -a(t^2 - 1) = x - at^2,$$

so that $x = a$; ie the reflected ray passes through the focus.

(3) Suppose that $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola. Show that $pq = -1$.

Solution

If the focus is S , the gradient of PS equals the gradient of QS ;

$$\text{ie } \frac{2ap-0}{ap^2-a} = \frac{2aq-0}{aq^2-a} \Rightarrow \frac{p}{p^2-1} = \frac{q}{q^2-1}$$

Treating this as a quadratic in p (for example),

$$qp^2 + p(1 - q^2) - q = 0$$

$$\Rightarrow \left(p + \frac{1}{q}\right)(qp - q^2) = 0$$

[as we trying to show that one root is $p = -\frac{1}{q}$]

$$\Rightarrow p = -\frac{1}{q} \text{ or } q(p - q) = 0$$

$q = 0$ can be rejected, as PQ wouldn't pass through S , and $p \neq q$, otherwise PQ is not a chord

(4) If the tangents to a parabola at P and Q are perpendicular, show that the chord PQ passes through the focus S of the parabola.

Solution

The gradients of the two tangents are $\frac{1}{p}$ and $\frac{1}{q}$ (standard result). As the tangents are perpendicular, $\left(\frac{1}{p}\right)\left(\frac{1}{q}\right) = -1$, so that $pq = -1$.

Gradient of PS = $\frac{2ap-0}{ap^2-a} = \frac{2p}{p^2-1}$, and the gradient of

$$QS = \frac{2aq-0}{aq^2-a} = \frac{2q}{q^2-1}$$

We wish to show that these gradients are the same; ie that

$$\frac{2p}{p^2-1} = \frac{2q}{q^2-1}$$

$$\text{LHS} = \frac{2(-\frac{1}{q})}{(-\frac{1}{q})^2-1} = \frac{2q}{-1+q^2} = \text{RHS}$$

(5) Find the cartesian equations of the parabolas with:

(i) focus (4,4) and directrix $y = 0$

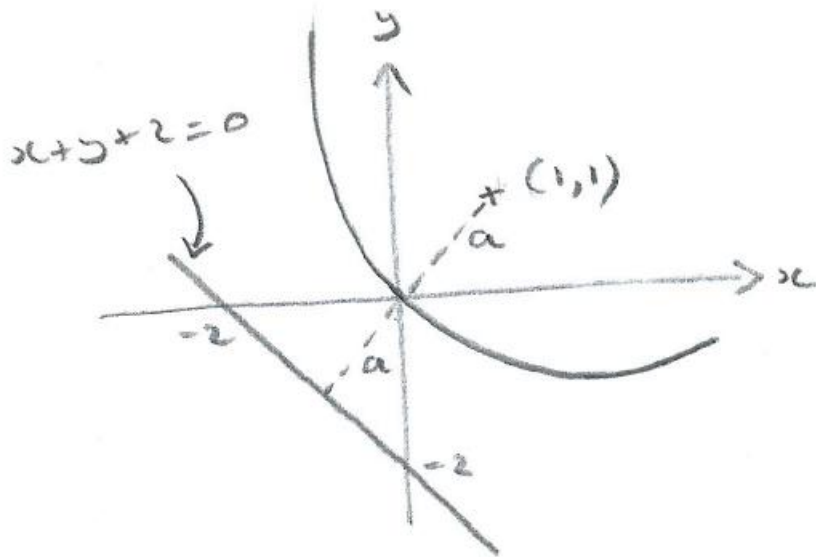
(ii) focus (1,1) and directrix $x + y + 2 = 0$

Solution

(i) The shortest distance from the focus to the directrix is $2a$, so $a = 2$.

Starting from the parabola with focus (0,2) and directrix $y = -2$ (with eq'n $x^2 = 4ay = 8y$), we need to make a translation of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, so that the eq'n becomes $(x - 4)^2 = 8(y - 2) = 8y - 16$

(ii) The directrix is $y = -x - 2$ (see diagram), and so $a = \sqrt{2}$



For general a , the parabola is obtained by rotating $y^2 = 4ax$ through 45° anti-clockwise.

Let the point (x, y) be transformed to the point (u, v) under the rotation.

$$\begin{aligned} \text{Then } \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} at^2 \\ 2at \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix} \end{aligned}$$

$$\text{Then } v - u = \frac{4at}{\sqrt{2}}, \text{ so that } v = \frac{at}{\sqrt{2}}(t + 2) = \frac{a(\frac{\sqrt{2}(v-u)}{4a})}{\sqrt{2}} \left(\frac{\sqrt{2}(v-u)}{4a} + 2 \right);$$

$$4v(4a) = \sqrt{2}(v - u)^2 + 8a(v - u);$$

$$\sqrt{2}(v - u)^2 = 8a(v + u);$$

$$(v - u)^2 = 4a\sqrt{2}(u + v)$$

$$\text{In this case, } (v - u)^2 = 8(u + v),$$

$$\text{which can be written as } (y - x)^2 = 8(x + y)$$

[When $t = 1$ (at P, say), we expect PS to be parallel to the directrix (where S is the focus), by comparison with $y^2 = 4ax$.

$$\text{When } t = 1, \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

and the gradient of PS is $\frac{3-1}{-1-1} = -1$, which is the gradient of the directrix.

(6) Suppose that $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola. Show that the tangents at P and Q meet on the directrix (the result of Exercise (A)(3) can be assumed).

Solution

The tangents have equations $py - x = ap^2$ and $qy - x = aq^2$ [standard result], and $pq = -1$, from Exercise (A)(3).

The tangents meet the directrix when $x = -a$, so that for the tangent at P, $py + a = ap^2$ and hence $y = ap - \frac{a}{p}$, and for the tangent at Q, $y = aq - \frac{a}{q} = a\left(-\frac{1}{p}\right) + ap = ap - \frac{a}{p}$ also.

Thus the tangents meet on the directrix.

(7) $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola. Show that the locus of the midpoint of PQ is a parabola, and establish its focus and directrix. (The result of Exercise (A)(3) can be assumed).

Solution

The midpoint of PQ is $\left(\frac{1}{2}a(p^2 + q^2), \frac{1}{2} \cdot 2a(p + q)\right)$

Thus, for a point (x, y) on the locus of the midpoint,

$$x = \frac{1}{2}a(p^2 + q^2) \text{ and } y = a(p + q)$$

$$\text{Then } y^2 = a^2(p^2 + q^2 + 2pq) = 2ax - 2a^2 = 4\left(\frac{a}{2}\right)(x - a)$$

This can be obtained from $y^2 = 4\left(\frac{a}{2}\right)x$ by a translation of $\begin{pmatrix} a \\ 0 \end{pmatrix}$, and so $y^2 = 4\left(\frac{a}{2}\right)(x - a)$ is a parabola with vertex $(a, 0)$ and focus

$\left(\frac{a}{2} + a, 0\right) = \left(\frac{3a}{2}, 0\right)$, and directrix $x = \frac{a}{2}$ [as the vertex is midway between the focus and the directrix]