Parabolas (6 pages; 30/10/15)

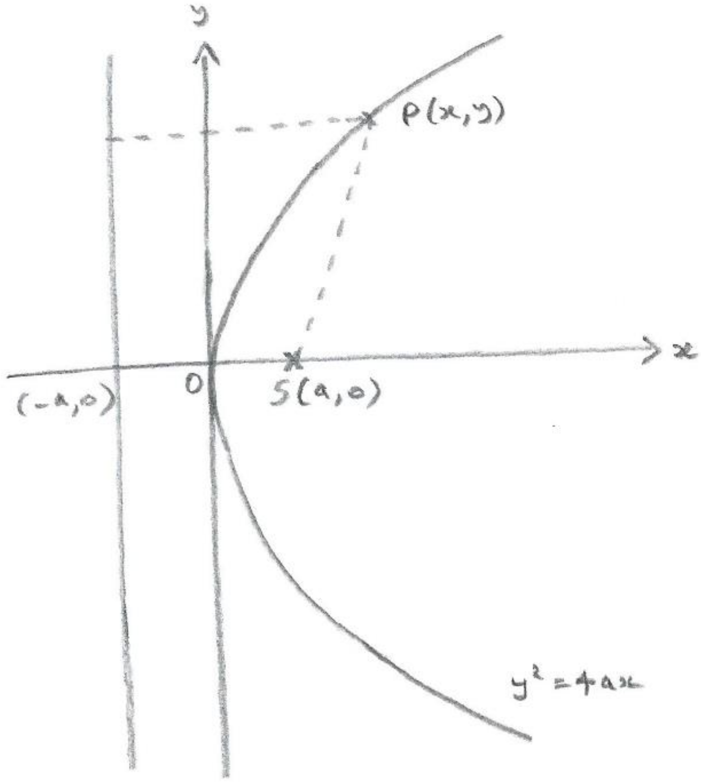
See "Conics" first, for features that are common to parabolas, ellipses and hyperbolas (as well as circles).

(1) Definitions

focus (S): (a,0) $(a > 0)$

directrix: $x = -a$

vertex: (0,0)



A parabola is the locus of points that are equidistant from the focus and the directrix.

The equation $y^2 = 4ax$ will be established in (2).

(2) Exercise

Given that the point (x, y) is equidistant from $(a, 0)$ and the line $x = -a$, show that $y^2 = 4ax$

Solution

Distance from $(a, 0)$ is $\sqrt{(x-a)^2 + (y-0)^2}$

Distance from the line $x = -a$ is $a + x$

So $\sqrt{(x-a)^2 + (y-0)^2} = a + x$

and hence $x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2$

so that $y^2 = 4ax$

(3) Parametric equations of a parabola

$x = at^2$, $y = 2at$ $(a > 0)$

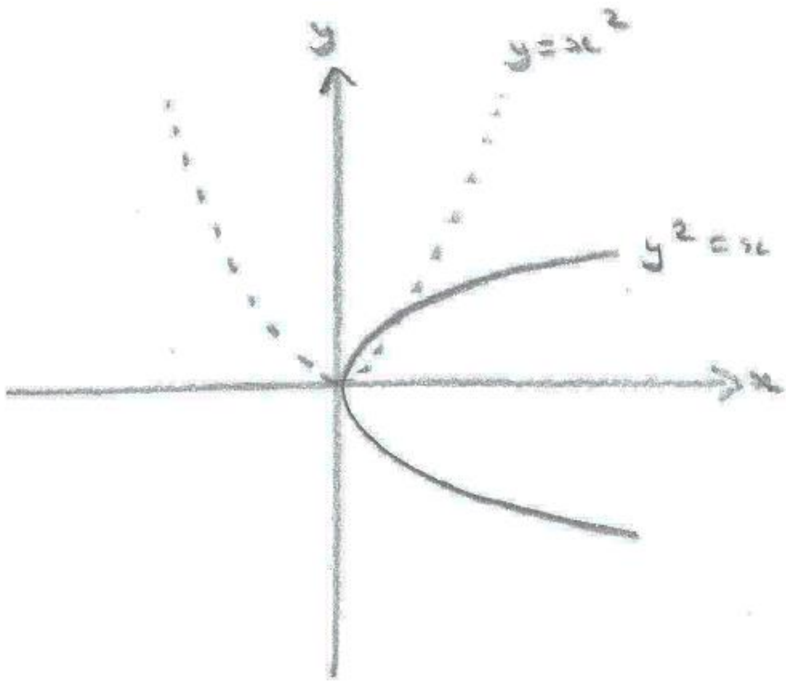
So general point is $(at^2, 2at)$

To confirm that the corresponding Cartesian equation is that of a parabola:

$y^2 = (2at)^2 = 4a^2t^2 = 4ax$

(4) Variations

Although it tends to be curves such as $y^2 = x$ (where $a = \frac{1}{4}$) that are usually cited as examples of parabolas, the inverse mapping $y = x^2$ is also a parabola.



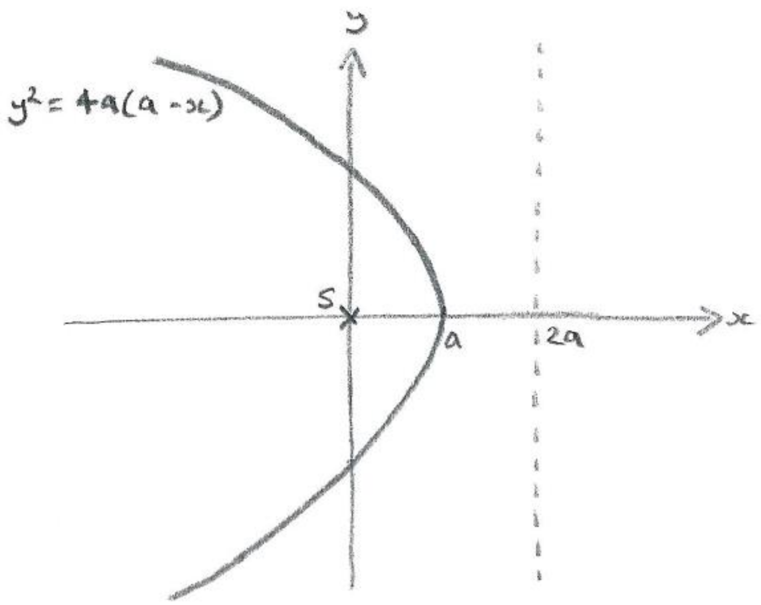
Other parabolas can be obtained by translating, stretching or rotating $y^2 = x$.

For example, $(y-b)^2 = x - a$ results from a translation of $\binom{a}{b}$, whilst $(2b-y)^2 = 2a - x$ represents a rotation of 180° about the point (a, b). [See separate notes on transformations.]

(5) Polar form

See "Conics" for the derivation of the polar form of a general conic: $r = \frac{ep}{1+ecos\theta}$, where p is the (positive) distance between the focus and the directrix, for the case where the directrix is vertical and lies to the right of the pole.

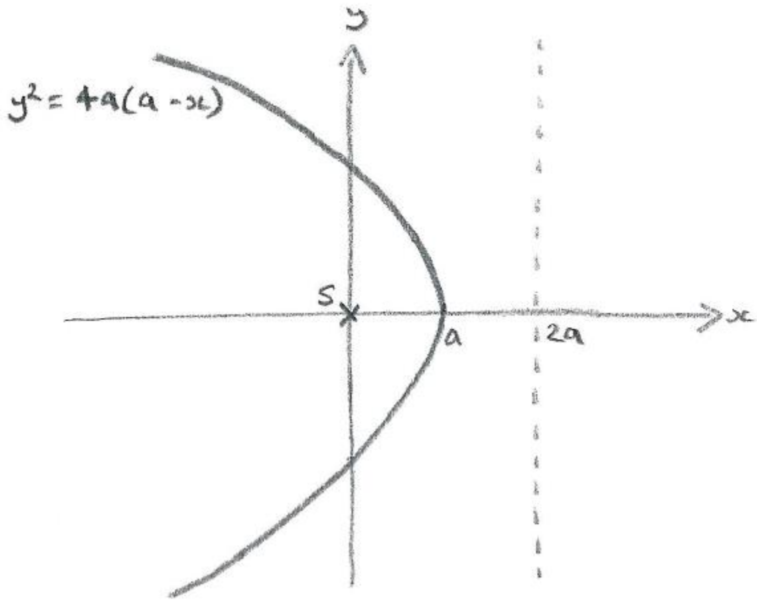
In order for the parabola to have the same location and orientation as the general conic used to derive the above formula, we reflect $y^2 = 4ax$ in the line $x = \frac{a}{2}$, to give $y^2 = 4a(a-x)$ [this can also be arrived at by translating by a to the left and then reflecting in the y-axis]. This means that the focus is moved to the Origin (see diagram below).



For this parabola, $e = 1$ and $p = 2a$, so that

$r = \frac{ep}{1+ecos\theta}$ becomes $r = \frac{2a}{1+cos\theta}$

(6) Reconciliation with the Cartesian form of the parabola



As $x = rcos\theta$ and $y = rsin\theta$,

$y^2 = 4a(a - x) \Rightarrow r^2sin^2\theta = 4a^2 - 4arcos\theta$

$\Rightarrow r^2sin^2\theta + 4arcos\theta = 4a^2$

$\Rightarrow (rsin\theta + 2acot\theta)^2 - 4a^2cot^2\theta = 4a^2$

$\Rightarrow (rsin\theta + 2acot\theta)^2 = 4a^2cosec^2\theta$ (A)

$\Rightarrow rsin\theta + 2acot\theta = 2acosec\theta$

$\Rightarrow rsin^2\theta + 2acos\theta = 2a$

$\Rightarrow r = \frac{2a(1-cos\theta)}{(1-cos^2\theta)} = \frac{2a}{1+cos\theta}$

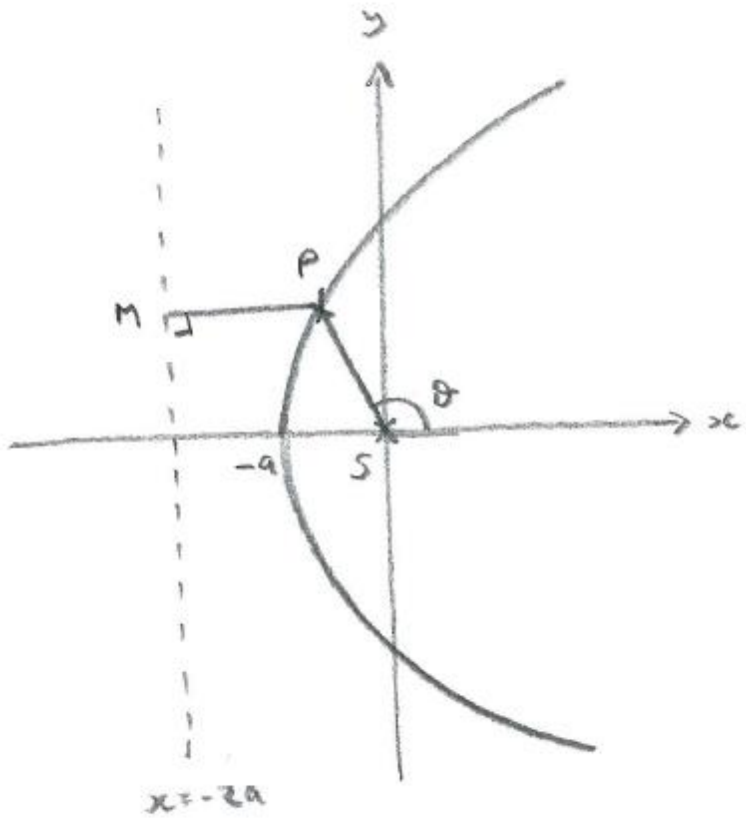
[Check: $\theta = 0 \Rightarrow r = a$ & $\theta = \frac{\pi}{2} \Rightarrow r = 2a$, which agrees with

$y^2 = 4a(a - x)$]

[The alternative solution of (A): $rsin\theta + 2acot\theta = -2acosec\theta$ is rejected, as it leads to $r = \frac{-2a}{1-cos\theta}$, which gives $r = -\infty$ when

$\theta = 0$]

Exercise: Find the appropriate polar form corresponding to the parabola shown below, and reconcile it with the Cartesian form.



The Cartesian form is $y^2 = 4a(x + a)$ (being $y^2 = 4ax$ translated a to the left).

Referring to the diagram above, $\frac{PS}{PM} = 1 \Rightarrow \frac{r}{2a - r\cos(\pi - \theta)} = 1$

so that $2a + r\cos\theta = r$ (as $\cos(\pi - \theta) = -\cos\theta$)

and hence $r = \frac{2a}{1 - \cos\theta}$

As $x = rcos\theta$ and $y = rsin\theta$,

$y^2 = 4a(x+a) \Rightarrow r^2sin^2\theta = 4arcos\theta + 4a^2$

$\Rightarrow r^2sin^2\theta - 4arcos\theta = 4a^2$

$\Rightarrow (rsin\theta - 2acot\theta)^2 - 4a^2cot^2\theta = 4a^2$

$\Rightarrow (rsin\theta - 2acot\theta)^2 = 4a^2cosec^2\theta$ (A)

$\Rightarrow rsin\theta - 2acot\theta = 2acosec\theta$

$\Rightarrow rsin^2\theta - 2acos\theta = 2a$

$\Rightarrow r = \frac{2a(1+cos\theta)}{(1-cos^2\theta)} = \frac{2a}{1-cos\theta}$

[Check: $\theta = 0 \Rightarrow r = \infty$ & $\theta = \frac{\pi}{2} \Rightarrow r = 2a$, which agrees with

$y^2 = 4a(x+a)$]