Parabolas Q5 [Problem/H](30/5/21)

Find the cartesian equations of the parabolas with:

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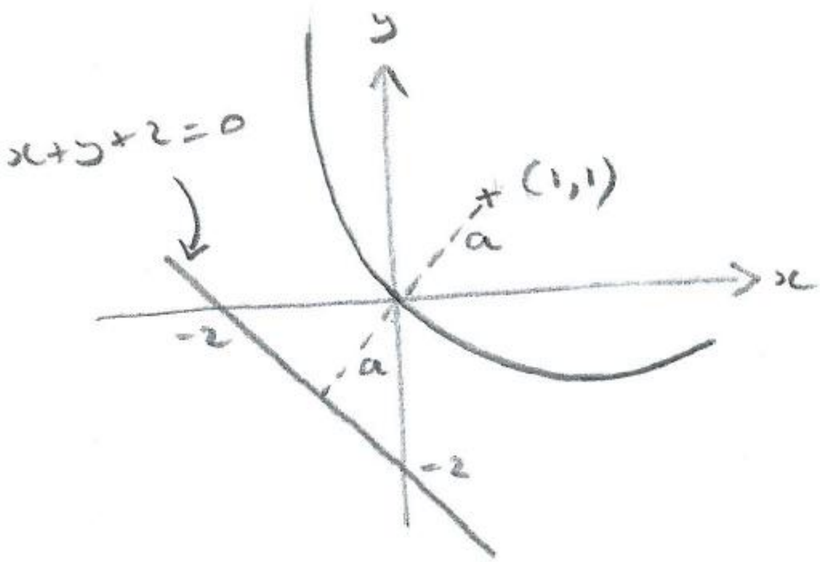
Solution

(i) The shortest distance from the focus to the directrix is $2a$, so $a = 2$.

Starting from the parabola with focus (0,2) and directrix $y = -2$ (with eq'n $x^2 = 4ay = 8y$), we need to make a translation of $\binom{4}{2}$, so that the eq'n becomes $(x-4)^2 = 8(y-2) = 8y - 16$

(ii) The directrix is $y = -x - 2$ (see diagram), and so $a = \sqrt{2}$



For general a, the parabola is obtained by rotating $y^2 = 4ax$ through 45° anti-clockwise.

Let the point (x, y) be transformed to the point (u, v) under the rotation.

Then $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} at^2 \\ 2at \end{pmatrix}$

$= \frac{1}{\sqrt{2}} \begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix}$

Then $v - u = \frac{4at}{\sqrt{2}}$, so that $v = \frac{at}{\sqrt{2}}(t + 2) = \frac{a(\frac{\sqrt{2}(v-u)}{4a})}{\sqrt{2}} \left(\frac{\sqrt{2}(v-u)}{4a} + 2 \right)$;

$4v(4a) = \sqrt{2}(v - u)^2 + 8a(v - u)$;

$\sqrt{2}(v - u)^2 = 8a(v + u)$;

$(v - u)^2 = 4a\sqrt{2}(u + v)$

In this case, $(v - u)^2 = 8(u + v)$,

which can be written as $(y - x)^2 = 8(x + y)$

[When $t = 1$ (at P, say), we expect PS to be parallel to the directrix (where S is the focus), by comparison with $y^2 = 4ax$.

When $t = 1$, $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} at^2 - 2at \\ at^2 + 2at \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$,

and the gradient of PS is $\frac{3-1}{-1-1} = -1$, which is the gradient of the directrix.