Parabolas Q3 [15 marks] (30/5/21)

Exam Boards

OCR : -

MEI: -

AQA: -

Edx: Further Pure 1 (Year 1)

Suppose that $P\ (ap^2, 2ap)$ and $Q\ (aq^2, 2aq)$ are two points on the parabola $y^2 = 4ax$, such that the chord PQ passes through the focus of the parabola.

(i) Show that $pq = -1$. [7 marks]

(ii) Show that the tangents at P and Q meet on the directrix.

[The equations of the tangents can be quoted without proof.]

[3 marks]

(iii) Show that the locus of the midpoint of PQ is a parabola, and establish its focus and directrix. [5 marks]

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Solution

(i) If the focus is S, the gradient of PS equals the gradient of QS

[1 mark]

ie $\frac{2ap-0}{ap^2-a} = \frac{2aq-0}{aq^2-a} \Rightarrow \frac{p}{p^2-1} = \frac{q}{q^2-1}$ [1 mark]

Treating this as a quadratic in p (for example),

$qp^2 + p(1-q^2) - q = 0$ [1 mark]

$\Rightarrow \left(p + \frac{1}{q}\right)(qp - q^2) = 0$ [1 mark]

[as we trying to show that one root is $p = -\frac{1}{q}$]

$\Rightarrow p = -\frac{1}{q}\ or\ q(p-q) = 0$ [1 mark]

$q = 0$ can be rejected, as PQ wouldn't pass through S, and $p \neq q$, otherwise PQ is not a chord;

hence $pq = -1$, as required [2 marks]

(ii) The tangents have equations $py - x = ap^2$

and $qy - x = aq^2$, and $pq = -1$, from (i).

The tangents meet the directrix when $x = -a$, so that for the tangent at P, $py + a = ap^2$ and hence $y = ap - \frac{a}{p}$, [1 mark]

and for the tangent at Q, $y = aq - \frac{a}{q} = a\left(-\frac{1}{p}\right) + ap = ap - \frac{a}{p}$ also. Thus the tangents meet on the directrix. [2 marks]

(iii) The midpoint of PQ is $\left(\frac{1}{2}a(p^2 + q^2), \frac{1}{2}.2a(p+q)\right)$ [1 mark]

Thus, for a point (x, y) on the locus of the midpoint,

$x = \frac{1}{2}a(p^2 + q^2)$ and $y = a(p + q)$

Then $y^2 = a^2(p^2 + q^2 + 2pq) = 2ax - 2a^2 = 4(\frac{a}{2})(x - a)$

[1 mark]

This can be obtained from $y^2 = 4(\frac{a}{2})x$ by a translation of $\binom{a}{0}$, and so $y^2 = 4(\frac{a}{2})(x - a)$ is a parabola with vertex $(a, 0)$ and focus $\left(\frac{a}{2} + a, 0\right) = (\frac{3a}{2}, 0)$, [2 marks]

and directrix $x = \frac{a}{2}$ [as the vertex is midway between the focus and the directrix] [1 mark]