Oscillations - Exercises (Sol'ns) (6 pages; 9/8/19)

(1) A lift has an elastic string suspended from its ceiling, with a mass of 10 grams at the end of the string. The string has natural length 80 cm, and modulus of elasticity $20N$. Initially, when the lift is stationary, the mass is hanging in equilibrium. The lift then starts to ascend with an acceleration of 0.2 ms^{-2}. Show that the extension of the string after t $secs$ is $0.4 - 0.008\cos(50t)$ cm.

[Assume that $g = 9.8ms^{-2}$]

Solution

Let x be the distance of the mass below the level of the ceiling of the lift when it is stationary, measured relative to the lift's surroundings.

Then $x = 0.8 + e - y$,

where e is the extension of the string and y is the distance moved (upwards) by the lift,

so that $\ddot{x} = \ddot{e} - \ddot{y} = \ddot{e} - 0.2$

Considering the forces on the mass,

$0.01g - T = 0.01\ddot{x}$,

where T is the tension in the string,

and by Hooke's law, $T = \frac{20}{0.8}e$

So $0.01g - \frac{20}{0.8}e = 0.01\ddot{x} = 0.01(\ddot{e} - 0.2)$,

and hence $g - 2500e = \ddot{e} - 0.2$,

or $\ddot{e} + 2500e = 9.8 + 0.2 = 10$ (*)

To solve the differential equation:

the auxiliary equation is $\lambda^2 + 2500 = 0$,

with roots $\lambda = \pm 50i$,

so that the complementary function is $Ae^{50it} + Be^{-50it}$

or $(A+B)cos50t + (A-B)isin50t$

or $Ccos50t + Dsin50t$,

which can be written as $Ecos(50t + \alpha)$

The particular integral of the differential equation is a constant F

(as the RHS of (*) is a constant),

such that $2500F = 10$, so that $F = 0.004$

Thus the general solution of (*) is:

$e = Ecos(50t + \alpha) + 0.004$ (**)

and $\dot{e} = -50Esin(50t + \alpha)$

When $t = 0$, and the mass is hanging in equilibrium,

$0.01g - T = 0$ and $T = \frac{20}{0.8}e$,

so that $0.01g = \frac{20}{0.8}e$ and $e = \frac{49}{12500}$

Also, at $t = 0$, $\dot{e} = 0$, so that $\alpha = 0$

Thus, from (**), $\frac{49}{12500} = E + 0.004$,

and $E = -0.00008$,

so that $e = 0.004 - 0.00008cos(50t)\ m$

or $0.4 - 0.008cos(50t)\ cm$

(2) A flexible bar is embedded horizontally in a wall. A particle rests on the free end of the bar, and the bar (with the particle) is pulled down 2cm below the horizontal, and then released. Given that the bar and particle start to perform simple harmonic motion about the horizontal position, with 5 cycles per second, how long is it before the particle loses contact with the bar, and what speed does it have at that point? [Note: The particle will not be in contact with the bar long enough to complete a cycle of the simple harmonic motion.]

Solution

Let x be the displacement of the bar from the horizontal, where upwards is the positive direction.

Then (converting to S.I. units), $x = 0.02sin(\omega t)$, if (for convenience) we measure time from when $x = 0$.

Then, as 1 cycle (ie 2π radians) takes $\frac{1}{5}$ sec.,

$\omega\left(\frac{1}{5}\right) = 2\pi$, so that $\omega = 10\pi$

and $\ddot{x} = -\omega^2 x$

The particle is subject to gravity and a reaction force $R(x)$ from the bar, so that

$R(x) - mg = m\ddot{x}$, whilst the particle is in contact with the bar

(where m is the mass of the particle)

The particle loses contact with the bar when $R(x) = 0$;

ie when $-mg = m\ddot{x}$ and $\ddot{x} = -g$

As $\ddot{x} = -\omega^2 x$, this occurs when $-\omega^2 x = -g$; ie $x = \frac{g}{\omega^2}$

(note that this is where $x > 0$; ie above the horizontal).

The bar and particle take a quarter of a cycle, ie $\frac{1}{20}$ sec., to travel from the point of release to the horizontal.

If contact is lost at T secs after the horizontal has been reached,

$\frac{g}{\omega^2} = x = 0.02sin(\omega T)$, where $\omega = 10\pi$.

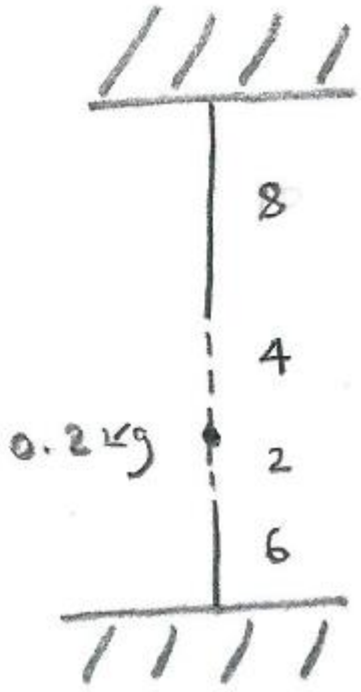
Then $T = \frac{1}{10\pi} sin^{-1}\left(\frac{9.8}{100\pi^2(0.02)}\right) = 0.016537$

and hence the required time from release is $\frac{1}{20} + 0.016537 =$ $0.066537 = 0.0665$ s (3sf)

The speed of the particle is $\dot{x}(T) = 0.02(10\pi)\cos(10\pi T)$

$= 0.54542\ ms^{-1}$ or $54.5cms^{-1}$(3sf)

(3) A 0.2 kg mass is held between two elastic strings, as shown in the diagram. The upper string has original length 8m and modulus of elasticity 2N, and is initially extended by 4m. The lower string has original length 6m and modulus of elasticity 1N, and is initially extended by 2m. When the mass is released, determine its subsequent motion (assume $g = 10$, and ignore any resistance to motion).



Solution

Let x be the displacement of the mass above its initial position.

By N2L, $(0.2)\ddot{x} = \frac{4-x}{8}(2) - \frac{2+x}{6}(1) - (0.2)(10)$

so that $\ddot{x} = 5\left(1 - \frac{1}{3} - 2\right) - 5x\left(\frac{1}{4} + \frac{1}{6}\right) = -\frac{20}{3} - \frac{25x}{12}$

Method 1

Writing $\ddot{x} + \frac{25x}{12} = -\frac{20}{3}$ gives a complementary function of

$x = Acos(\frac{5}{\sqrt{12}}t + \alpha)$

and a trial function of $x = C$ for the particular integral gives

$\frac{25C}{12} = -\frac{20}{3}$, so that $C = -\frac{16}{5}$

and the general solution is $x = Acos\left(\frac{5}{\sqrt{12}}t + \alpha\right) - \frac{16}{5}$

so that $\dot{x} = -\frac{5}{\sqrt{12}}Asin\left(\frac{5}{\sqrt{12}}t + \alpha\right)$

Then $t = 0, \dot{x} = 0 \Rightarrow \alpha = 0$

and $t = 0, x = 0 \Rightarrow 0 = A - \frac{16}{5}$

so that the particular solution is $x = \frac{16}{5}[cos\left(\frac{5}{\sqrt{12}}t\right) - 1]$

This is SHM of amplitude $\frac{16}{5}$ about $x = -\frac{16}{5}$

The period of oscillations is given by $\frac{5}{\sqrt{12}}T = 2\pi$,

and is $T = \frac{2\pi\sqrt{12}}{5} = 4.35$ s (3sf)

Thus the mass falls when released, and rises again to its initial position (as would be predicted by conservation of energy).

Method 2

$\ddot{x} = -\frac{20}{3} - \frac{25x}{12}$

Let $-\frac{20}{3} - \frac{25x}{12} = -\frac{25y}{12}$, so that $y = x + \frac{20}{3}\left(\frac{12}{25}\right) = x + \frac{16}{5}$

Then $\ddot{y} = \ddot{x} = -\frac{25y}{12}$

which gives SHM of amplitude $\frac{16}{5}$ about $y = 0$; ie $x = -\frac{16}{5}$