Numerical Integration - Midpoint & Trapezium rules

(7 pages; 22/10/18)

See also:

"Numerical Integration - Simpson's rule"

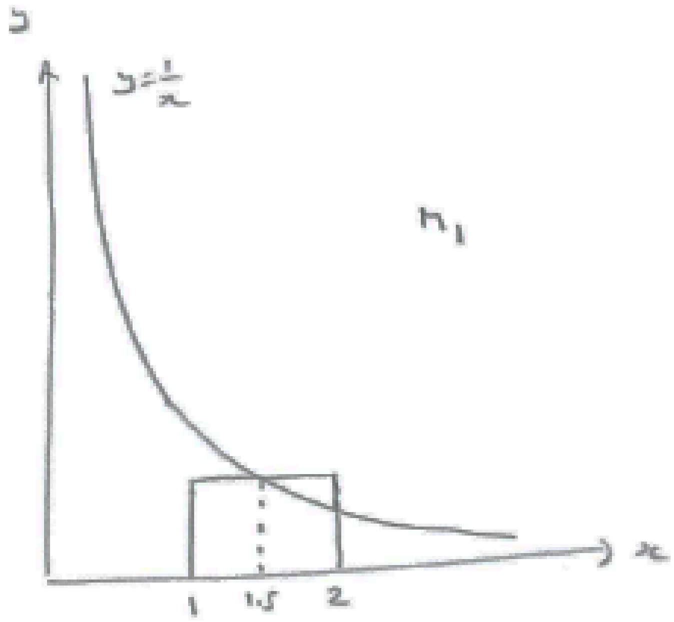
"Numerical Integration - Convergence"

(A) Midpoint rule (aka the mid-ordinate rule)

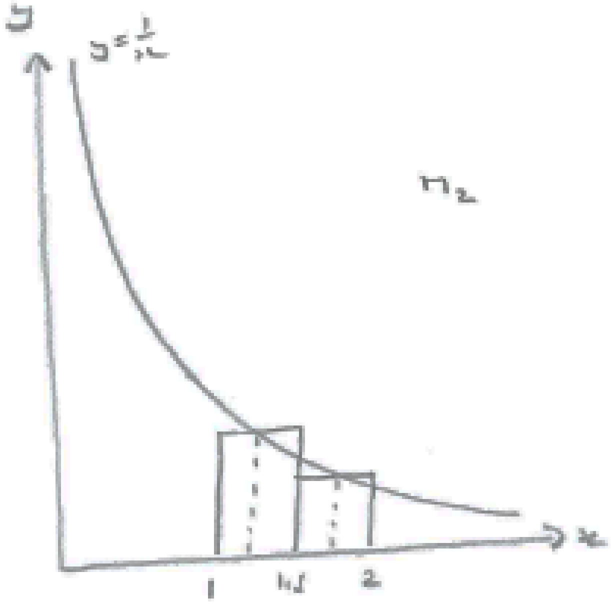
(1) M_n approximates the area under the curve by the sum of the areas of n rectangles, where the height of each rectangle is determined by the value of the function at the midpoint of the rectangle. This is shown below for $n = 1, 2$ & 3.

Example: $\int_1^2 \frac{1}{x}\, dx$

Exact value $= [lnx]_1^2 = ln2 - ln1 = ln2 = 0.69315$ (5dp)



$M_1 = \frac{1}{1.5} \times 1 = 0.66667$



$M_2 = \left(\frac{1}{1.25} + \frac{1}{1.75}\right) \times 0.5 = 0.68571$

$M_3 = \left(\frac{1}{7/6} + \frac{1}{9/6} + \frac{1}{11/6}\right) \times \frac{1}{3} = 0.68975$

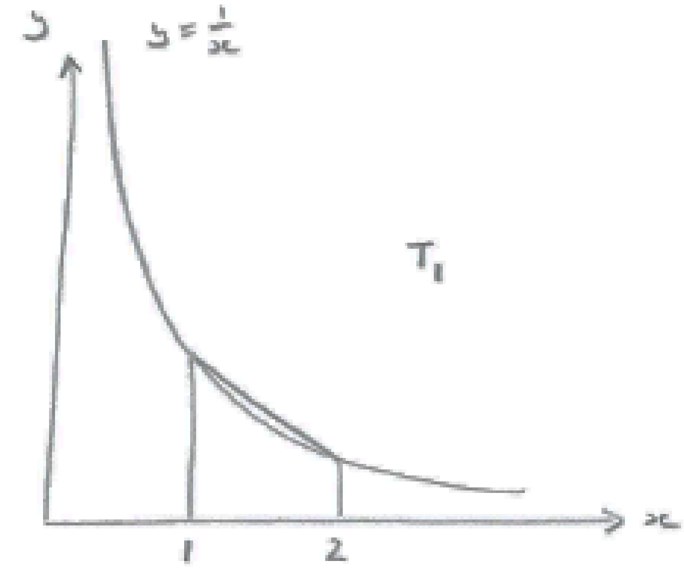
(2) The general formula is

$M_n = h(f_{0.5} + f_{1.5} + \ldots + f_{n-0.5})$,

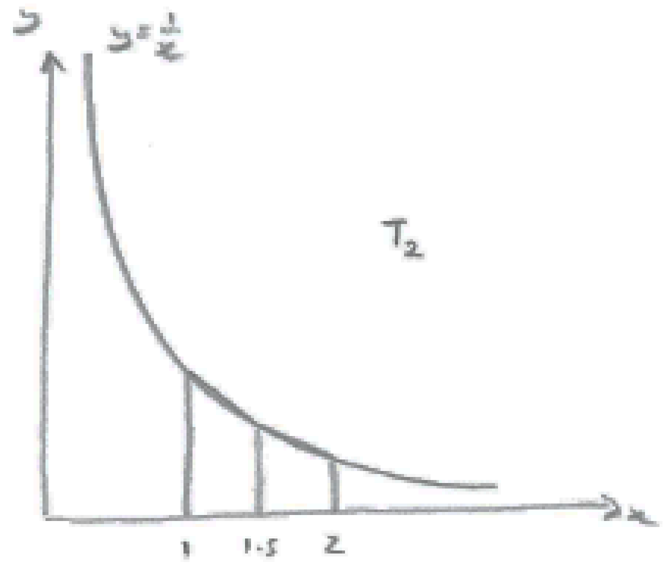
where h is the width of each rectangle.

(B) Trapezium rule

(1) T_n approximates the area under the curve by the sum of the areas of n trapezia, where the sides of each trapezium are the values of the function. This is shown below for $n = 1, 2$ & 3 (for the same example as for the midpoint rule, where the exact value was 0.69315).



$T_1 = \frac{1}{2}\ \left(\frac{1}{1} + \ \frac{1}{2}\ \right) \times 1 = 0.75$



$T_2 = \frac{1}{2}(\frac{1}{1} + \frac{1}{1.5}) \times 0.5 + \frac{1}{2}(\frac{1}{1.5} + \frac{1}{2}) \times 0.5$

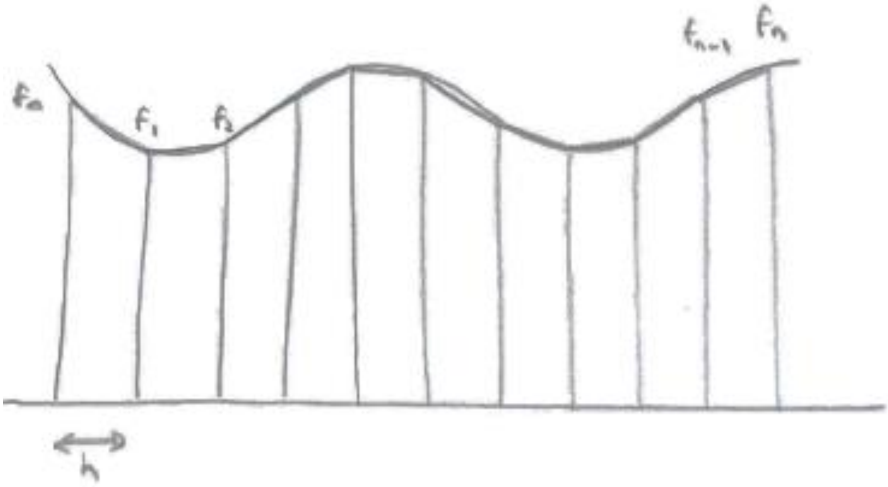
$= \frac{1}{2}\ [\frac{1}{1} + \ \frac{1}{2} + 2\left(\frac{1}{1.5}\right)] = 0.70833$

$T_3 = \frac{1}{2}\,(\frac{1}{1} + \ \frac{1}{2} + 2\left(\frac{1}{4/3} + \frac{1}{5/3}\right)) \times \frac{1}{3} = 0.7$

(2) The general formula is

$T_n = \frac{1}{2}h\,\{\, f_0 + f_n + 2(f_1 + f_2 + \cdots + f_{n-1})\},$

where h is the width of each trapezium.



(C) Connections between Trapezium and Midpoint rules

(1) It can be seen that $T_{2n} = ½\left(T_n + \ M_n\right)$

Proof

Let W be the width of the interval being considered.

As $T_n = \frac{1}{2}h\,\{\, f_0 + f_n + 2(f_1 + f_2 + \cdots + f_{n-1})\},$

$T_{2n} = \frac{1}{2}\left(\frac{W}{2n}\right)\{\, f_0 + f_{2n} + 2(f_1 + f_2 + \cdots + f_{2n-1})\}$

(noting that f_1, f_2 etc are now defined differently).

Using the definitions of f_1, f_2 that appear in T_{2n},

T_n becomes $\frac{1}{2}\left(\frac{W}{n}\right)\{\, f_0 + f_{2n} + 2(f_2 + f_4 + \cdots + f_{2n-2})\}.$

Also, M_n becomes $\left(\frac{W}{n}\right)(f_1 + f_3 + \ldots + f_{2n-1})$,

and so $T_n + M_n = \frac{1}{2}\left(\frac{W}{n}\right)\{f_0 + f_{2n} + + 2(f_2 + f_4 + \cdots + f_{2n-2})$

$$+2(f_1 + f_3 + \ldots + f_{2n-1})\}$$

$= 2T_{2n}$, as required.

(2) Comparison of values from Trapezium & Midpoint rules

n	T_n	M_n	$\frac{T_n - A}{A} \times 100\%$	$\frac{M_n - A}{A} \times 100\%$
A(actual)	0.69315	0.69315		
1	0.75	0.66667	8.202%	−3.820%
2	0.70833	0.68571	2.190%	−1.073%
3	0.7	0.68975	0.988%	−0.491%

Notes

(i) For this example, the T_n fall towards the actual value, whilst the M_n rise. In general, either T_n or M_n may fall (depending on whether the curve is convex or concave*), and the other will do the opposite.

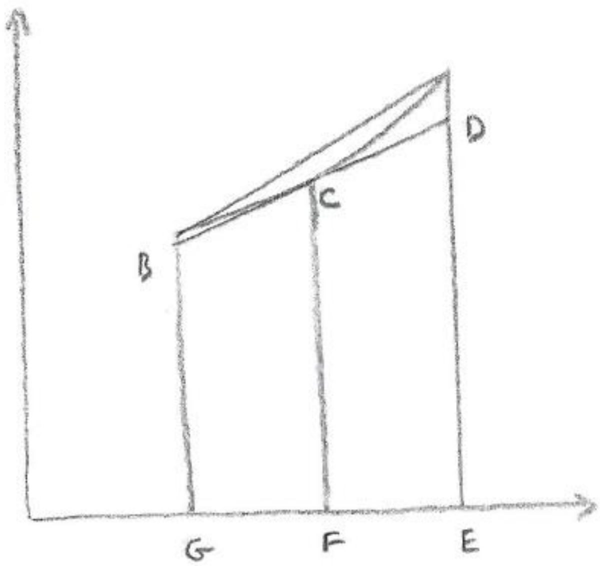
(ii) The absolute size of the relative error for T_n

(ie $\left|\frac{T_n - A}{A} \times 100\right|$) is approximately twice that for M_n

(ie $\left|\frac{M_n - A}{A} \times 100\right|$). And also $|T_n - A| \approx |M_n - A|$ (see Appendix).

* A convex function is one such as $y = e^x$ (think of conve^x) or $y = \frac{1}{x}$, whilst a concave function is one such as $y = -x^2$. (See "Convexity and concavity" note.)

(3) Upper and lower bounds



For a convex function (as above), $M_n < A < T_n$,

where A is the actual value of the area under the curve.

$A < T_n$ is clear from the diagram

To see why $M_n < A$, we draw the tangent to the curve at the midpoint C (the line BCD in the diagram). As the function is convex, the curve lies above the tangent on both sides of C, so that the trapezium BCDEFG has an area smaller than A. This trapezium has the same area as the rectangle in the mid-point rule, with base GE and height FC.

Similarly for a concave function, $T_n < A < M_n$.

In practice, a curve may need to be split up into convex and concave parts.

Appendix: Demonstration that $|T_n - A| \approx |M_n - A|$

Consider (for example) the concave case, where $T_n < A < M_n$.

Result to prove: $A - T_n \approx 2(M_n - A)$

$T_{2n} = ½\,(T_n + M_n)$ and $T_{2n} - A \approx \frac{1}{4}(T_n - A)$

so that $= ½\left(T_n + M_n\right) - A \approx \frac{1}{4}(T_n - A)$

$\Rightarrow\ 2T_n + 2M_n - 4A \approx T_n - A$

$\Rightarrow\ T_n - A \approx 2A - 2M_n$

$\Rightarrow\ A - T_n \approx 2(M_n - A)$