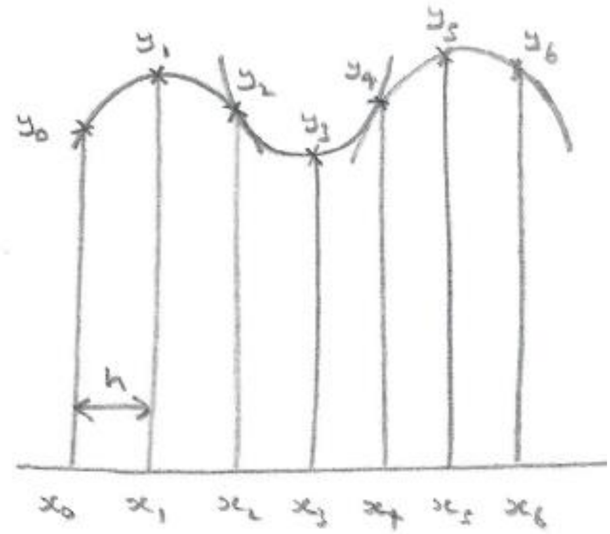
Numerical Integration - Simpson's rule

(7 pages; 22/10/18)

See also:

"Numerical Integration - Convergence"

(1) Given a number of points on the curve, Simpson's rule is derived by fitting a series of overlapping quadratic curves to the points. (Note that, unless they lie on a straight line, a quadratic curve can be found that passes through any 3 points.)



[Note: The diagram doesn't show the original curve; just the points on it and the quadratic curves fitted to those points.]

Referring to the diagram above, if $y_0, y_1, \dots, y_6$ are the given y values (or 'ordinates'), then the first quadratic curve is fitted to the points (x_0, y_0), (x_1, y_1) & (x_2, y_2); the second quadratic curve is fitted to the points (x_2, y_2), (x_3, y_3) & (x_4, y_4) , and so on.

Integration is then used to find the areas under the quadratic curves, and these are added to give the final formula, which for the above example is:

$S_6 = \frac{h}{3}\left(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right)$

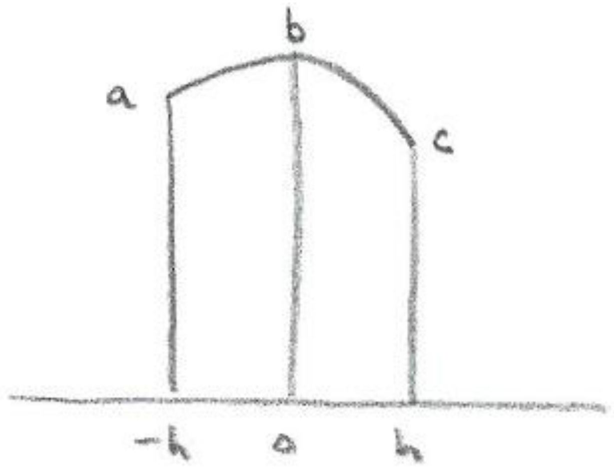
(to be proved shortly)

Notes

(i) This is the notation that is now used in the MEI exams. The 6 in S_6 is the number of strips. In an alternative notation, S_3 is written instead of S_6 (there being 3 quadratic curves for which areas are found). For Edexcel and AQA, it would seem that the S_n and S_{2n} notations are avoided (instead they say, for example, "Use Simpson's rule with 8 strips").

(ii) The quadratic functions can be found by either Newton's Forward Difference method or Lagrange's method (see separate notes).

(2) Proof



Let the quadratic function in the above diagram be

$y = px^2 + qx + b$

The area under the curve is $\int_{-h}^{h} px^2 + qx + b \, dx$

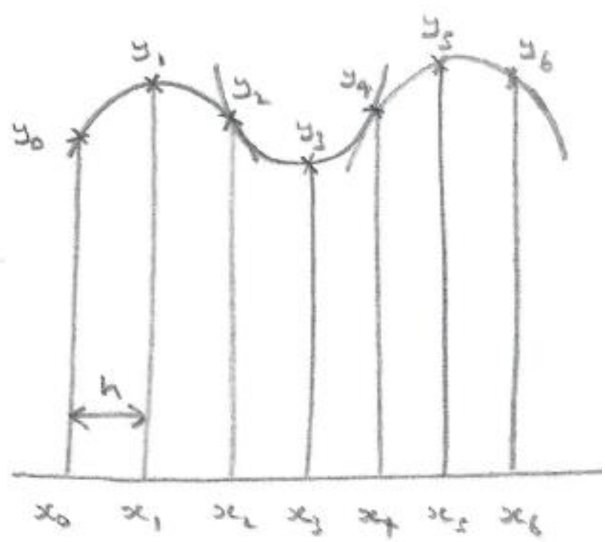
$$= \left[\frac{px^3}{3} + \frac{qx^2}{2} + bx\right]_{-h}^{h} = \frac{2ph^3}{3} + 2bh \quad (1)$$

Also $ph^2 - qh + b = a$ (2) and $ph^2 + qh + b = c$ (3)

Adding (1) & (2) gives $2ph^2 + 2b = a + c$ (4)

Then from (1) & (4),

Area $= \frac{h}{3}(a + c - 2b + 6b) = \frac{h}{3}(a + 4b + c)$



For 3 quadratic curves, the area is therefore

$$S_6 = \frac{h}{3}([y_0 + 4y_1 + y_2] + [y_2 + 4y_3 + y_4] + [y_4 + 4y_5 + y_6])$$

$= \frac{h}{3}\left(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right)$, as required

(3) This extends to the following general formula:

$$S_{2n} = \frac{1}{3}h\{ f_0 + f_{2n} + 4(f_1 + f_3 + \cdots + f_{2n-1})$$

$+2(f_2 + f_4 + \cdots + f_{2n-2})\}$,

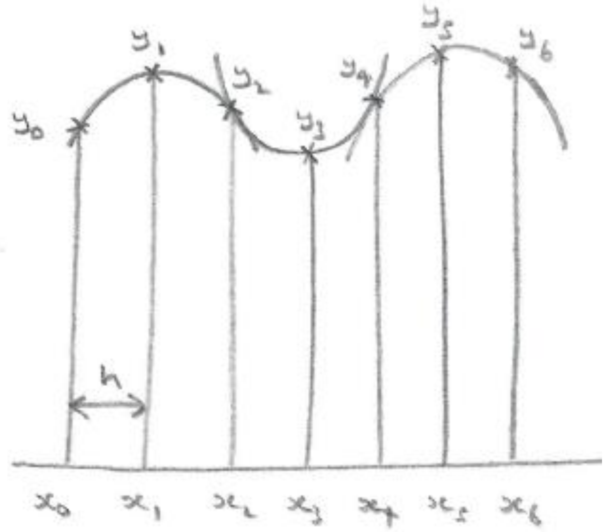
where the odd ordinates are multiplied by 4, and the even ones (except the first and last) are multiplied by 2.

(4) Formula for Simpson's rule in terms of the Trapezium & Midpoint rules

We will show that $S_{2n} = \frac{2M_n + T_n}{3}$.

This result is not approximate. In practice, it will usually be more convenient to find S_{2n} in this way, rather than from the formula derived in (2).

Demonstration for S_6:



We are dealing with 3 trapezia and 3 rectangles here, with bases of $2h$:

$T_3 = \frac{2h}{2}(y_0 + y_6 + 2(y_2 + y_4))$

$M_3 = 2h(y_1 + y_3 + y_5)$

So $\frac{2M_3+T_3}{3} = \frac{h}{3}(4(y_1 + y_3 + y_5) + (y_0 + y_6 + 2(y_2 + y_4)))$

	A	B	C	D
1				
2	n	T_n	M_n	$S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$
3				
4	1	0.75	0.66667	0.69445
5	2	0.70833	0.68571	0.69325
6	3	0.7	0.68975	0.69317
7				
8	Exact	0.69315	0.69315	0.69315

$= \frac{h}{3}\left(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right)$, as required.

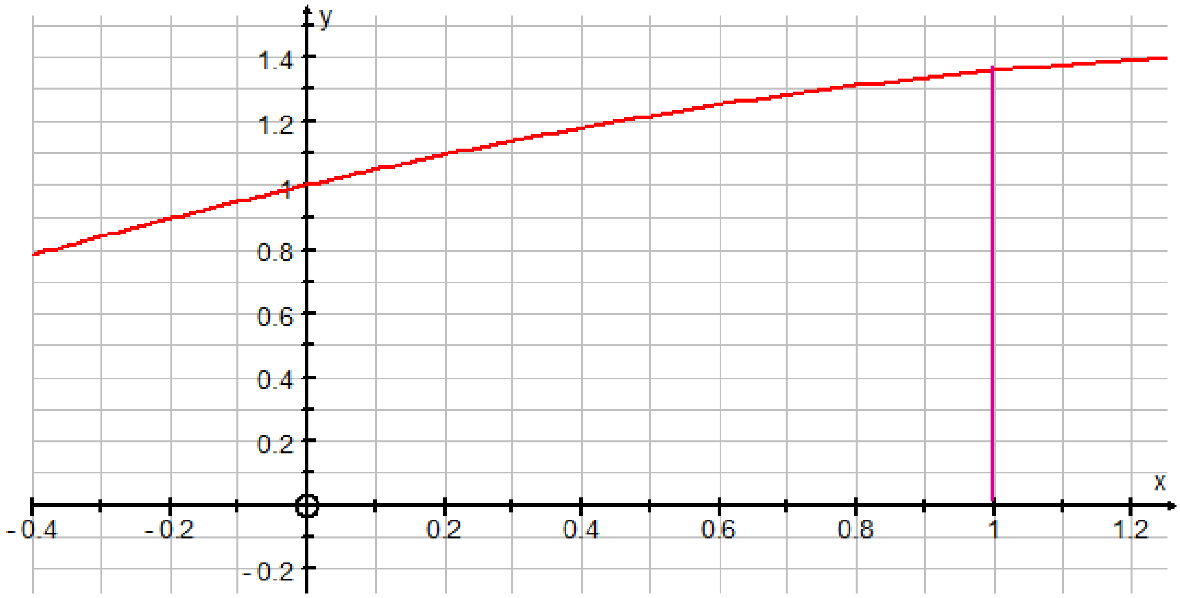
The table below applies this relation to a specific example (see "Midpoint & Trapezium rules"), and shows that the Simpson's rule estimate is much closer to the exact value.

(5) Relation between Simpson's rule and the Trapezium rule

$S_{2n} = \frac{2M_n+T_n}{3}$ and $T_{2n} = ½\,(T_n + M_n)$

$\Rightarrow S_{2n} = \frac{2(2T_{2n}-T_n)+T_n}{3} = \frac{4T_{2n}-T_n}{3}$

(6) Example: $\int_0^1 \sqrt{sinx + 1}\, dx$



$M_2 = 0.5\big(f(0.25) + f(0.75)\,\big)$

$f(0.25) = \sqrt{\sin(0.25) + 1} = 1.11687$

$f(0.75) = \sqrt{\sin(0.75) + 1} = 1.29678$

$M_2 = 0.5(1.11687 + 1.29678) = 1.20683$

$T_n = \frac{1}{2}h\{\mathrm{f}_0 + f_n + 2(f_1 + f_2 + \cdots + f_{n-1})\}$

$T_2 = \frac{1}{2}(0.5)\{f(0) + f(1) + 2f(0.5)\}$

$f(0) = \sqrt{\sin(0) + 1} = 1$

$f(0.5) = \sqrt{\sin(0.5) + 1} = 1.21632$

$f(1) = \sqrt{\sin(1) + 1} = 1.35701$

$T_2 = \frac{1}{2}(0.5)\{1 + 1.35701 + 2(1.21632)\} = 1.19741$

$T_{2n} = \frac{1}{2}(T_n + \ M_n)$

$T_4 = \frac{1}{2}(T_2 + \ M_2) = \frac{1}{2}(1.19741 + 1.20683) = 1.20212$

$M_4 = 0.25\big(f(0.125) + f(0.375) + f(0.625) + f(0.875\ \big))$

$= 0.25(1.06051 + 1.16888 + 1.25901 + 1.32949)$

$= 1.20447$

$S_{2n} = \frac{2M_n + T_n}{3}$

$S_4 = \frac{2M_2 + T_2}{3} = \frac{2\left(1.20683\ \right) + 1.19741}{3} = 1.20369$

$S_8 = \frac{2M_4 + T_4}{3} = \frac{2(1.20447) + 1.20212}{3} = 1.20369$