Numerical Integration - Convergence

(5 pages; 22/10/18)

[Note: Because the expression 'absolute error' is now used to mean $x - A$ (where x is the observed value and A is the actual (ie true) value), as opposed to the 'relative error' $\frac{x-A}{A}$), I will use 'absolute size' of y to indicate $|y|$. However, this isn't a standard expression.]

(1) Errors in T_n and M_n

In "Numerical Integration - Midpoint & Trapezium rules", it was seen that the absolute size of the relative error for T_n was approximately twice that for M_n. This is consistent with the formula

$S_{2n} = \frac{2M_n+T_n}{3}$: S_{2n} is more accurate than the other two methods (with the quadratic function producing a better fit than the straight line of the Trapezium method, for example), and can therefore be taken to be approximately equal to A, the actual value. Then, from the formula, A is approximately the given weighted average of M_n and T_n (see diagram below), with

$T_n - A \approx 2(A - M_n)$ (if $T_n > M_n$).

To summarise, S_{2n} is more accurate than M_n, and M_n is more accurate than T_n.

[Also, as seen in "Numerical Integration - Midpoint & Trapezium rules", for a convex function: $M_n < A < T_n$, and for a concave function: $T_n < A < M_n$.]

(2) Speed of convergence

For the Midpoint & Trapezium rules, it can be shown that the absolute size of the error (ie $|M_n - A|$ or $|T_n - A|$) is approximately proportional to h^2.

The Midpoint & Trapezium rules are accordingly described as '2nd order' methods.

So the absolute size of the error in $M_n \approx \lambda h^2$ and the absolute size of the error in $M_{2n} \approx \lambda \left(\frac{h}{2}\right)^2$ (and similarly for the Trapezium rule).

Hence $\frac{absolute\ size\ of\ error\ in\ M_{2n}}{absolute\ size\ of\ error\ in\ M_n} \approx \frac{\lambda\left(\frac{h}{2}\right)^2}{\lambda h^2} = \frac{1}{4}$

and the absolute size of error in $M_{2n} \approx \frac{1}{4} \times$ absolute size of error in M_n, which means that the Midpoint & Trapezium rules both have '1st order convergence'.

[This is where error $= k \times$ previous error; 2nd order convergence is where error $= k \times (\text{previous error})^2$]

Note the confusing terminology: 2nd order method, but 1st order convergence. (Note also that the proportionality constants λ and k are not the same quantity!)

Also, it can be shown that, for Simpson's rule:

absolute size of error in $S_{2n} \approx \lambda h^4$

so that $\frac{absolute\ size\ of\ error\ in\ S_{4n}}{absolute\ size\ of\ error\ in\ S_{2n}} \approx \frac{\lambda\left(\frac{h}{2}\right)^4}{\lambda h^4} = \frac{1}{16}$

Simpson's rule is a 4th order method, but again having 1st order convergence, as

$absolute\ size\ of\ error\ in\ S_{4n}$

$\approx \frac{1}{16} \times absolute\ size\ of\ error\ in\ S_{2n}$

(3) Finding *k*

It was shown in "Convergence - Introduction" that if a sequence x_r converges on α and has 1st order convergence, so that $e_r \approx ke_{r-1}$, where $e_r = x_r - \alpha$, then $\frac{x_{r+1}-x_r}{x_r-x_{r-1}} \approx k$.

This can be applied to M_n (as well as T_n and S_{2n}). Here the sequence is $M_1, M_2, M_4, \ldots$ (ie $x_3 = M_4$) and

$k = \frac{absolute\ size\ of\ error\ in\ M_{2n}}{absolute\ size\ of\ error\ in\ M_n} \approx \frac{1}{4}$, so that the 'ratios of differences'

$\frac{M_4-M_2}{M_2-M_1}, \frac{M_8-M_4}{M_4-M_2}, \frac{M_{16}-M_8}{M_8-M_4} \ldots \approx \frac{1}{4}$

The same result applies to T_n as well ,

as $k = \frac{absolute\ size\ of\ error\ in\ T_{2n}}{absolute\ size\ of\ error\ in\ T_n} \approx \frac{1}{4}$ also.

And a similar result for S_{2n} follows from the fact that

$k = \frac{absolute\ size\ of\ error\ in\ S_{4n}}{absolute\ size\ of\ error\ in\ S_{2n}} \approx \frac{1}{16}$; so $\frac{S_8-S_4}{S_4-S_2}$ etc $\approx \frac{1}{16} = 0.0625$

(4) Obtaining a better estimate of the integral by extrapolation

It is possible to extrapolate from a series of M_1, M_2, M_4 ...

by using the fact that successive ratios of differences

$\frac{M_4-M_2}{M_2-M_1}, \frac{M_8-M_4}{M_4-M_2}, \frac{M_{16}-M_8}{M_8-M_4}$ are known to be $\approx \frac{1}{4}$,

and similarly for the Trapezium and Simpson rules.

So, for example, $T_{32} - T_{16} \approx \frac{1}{4}(T_{16} - T_8)$

and hence $T_{32} \approx T_{16} + \frac{1}{4}(T_{16} - T_8)$

Similarly $T_{64} \approx T_{32} + \frac{1}{4}(T_{32} - T_{16})$

So $T_{64} \approx T_{16} + \frac{1}{4}(T_{16} - T_8) + \frac{1}{4^2}(T_{16} - T_8)$

And $T_{128} \approx T_{64} + \frac{1}{4}(T_{64} - T_{32})$

$\approx T_{16} + \frac{1}{4}(T_{16} - T_8) + \frac{1}{4^2}(T_{16} - T_8) + \frac{1}{4^2}(T_{32} - T_{16})$

$\approx T_{16} + \frac{1}{4}(T_{16} - T_8) + \frac{1}{4^2}(T_{16} - T_8) + \frac{1}{4^3}(T_{16} - T_8)$

So, by repeating this process,

$A \approx T_{16} + \frac{1}{4}(T_{16} - T_8) + \frac{1}{4^2}(T_{16} - T_8) + \frac{1}{4^3}(T_{16} - T_8) + \cdots$

$\approx T_{16} + \frac{1}{4}(T_{16} - T_8)\frac{1}{1-\frac{1}{4}}$

$\approx T_{16} + \frac{1}{3}(T_{16} - T_8)$

or $T_{32} + \frac{1}{3}(T_{32} - T_{16})$ etc

(with the same result applying for the Midpoint rule).

[Note: Starting from $T_{2n} - A \approx \frac{T_n - A}{4}, T_{2n} - \frac{T_n}{4} \approx \frac{3A}{4}$

$\Rightarrow A \approx \frac{4T_{2n}}{3} - \frac{T_n}{3}$ (ie the same result as above)]

Naturally we would use the best pair of T_n & T_{n-1} available (ie with the largest n).

Similarly,

$$A \approx S_{32} + \frac{1}{16}\,(S_{32} - S_{16})\left(\frac{1}{1-\frac{1}{16}}\right) = S_{32} + \frac{1}{15}\,(S_{32} - S_{16})\text{ etc}$$

Appendix: Alternative proof that $\frac{M_{2n}-A}{M_n-A} \approx \frac{1}{4} \Rightarrow \frac{M_{4n}-M_{2n}}{M_{2n}-M_n} \approx \frac{1}{4}$

$$\frac{M_{4n}-M_{2n}}{M_{2n}-M_n} \approx \frac{(M_{4n}-A)+(A-M_{2n})}{(M_{2n}-A)+(A-M_n)} = \frac{\frac{1}{4}(M_{2n}-A)-\frac{1}{4}(M_n-A)}{(M_{2n}-A)-(M_n-A)} = \frac{1}{4}$$