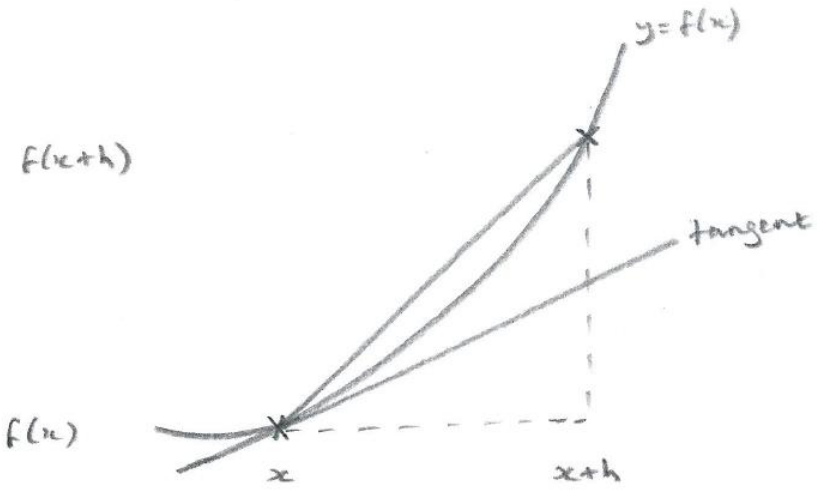
Numerical Differentiation (3 pages; 22/10/18)

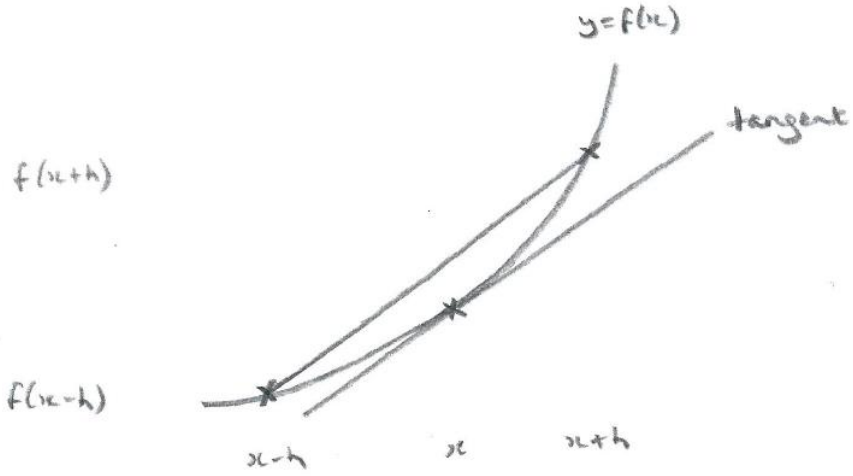
(1) Forward difference approximation to the derivative:

$f'(x) \approx \frac{f(x+h)-f(x)}{h}$ (for small h)



(2) Central difference approximation to the derivative:

$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$



(3) Example: $f(x) = sinx$

To find $f'(1)$ using the forward difference and central difference methods:

	A	B	C
1		$\frac{\sin(1+h)-\sin(1)}{h}$	$\frac{\sin(1+h)-\sin(1-h)}{2h}$
2	h		
3			
4	0.1	0.497363753	0.539402252
5	0.01	0.536085981	0.540293301
6	0.001	0.53988148	0.540302216
7	0.0001	0.540260231	0.540302305
8	0.00001	0.540298099	0.540302306
9	0.000001	0.540301885	0.540302306
10	0.0000001	0.540302264	0.540302306
11	0.00000001	0.540302303	0.540302308

[Actual value: $f'(1) = \cos(1) = 0.540302306$]

(4) The error for the forward difference can be shown to be proportional to h; ie it is a "1st order" method. The error for the central difference method can be shown to be proportional to h^2; ie it is a "2nd order" method.

As for Numerical Integration (see separate note), both methods have "1st order" convergence. The ratio of differences is $\frac{1}{2}$ for the Forward difference method (when h is halved), and $\frac{1}{4}$ for the Central difference method (by the same reasoning as for the Numerical Integration methods).

The Central difference method is usually an improvement on the Forward difference method.

(5) Using the ratio of differences to find an improved estimate via the Central difference method ,

$\frac{cd\left(\frac{h}{4}\right)-cd(\frac{h}{2})}{cd\left(\frac{h}{2}\right)-cd(h)} \approx 0.25$ (as k was found to be 0.25)

$\Rightarrow cd\left(\frac{h}{4}\right) - cd(\frac{h}{2}) \approx 0.25(cd\left(\frac{h}{2}\right) - cd(h))$

$\Rightarrow cd\left(\frac{h}{4}\right) \approx cd\left(\frac{h}{2}\right) + 0.25(cd\left(\frac{h}{2}\right) - cd(h))$

As for the Midpoint & Trapezium methods (see separate note),

limit of $cd \approx cd\left(\frac{h}{2}\right) + 0.25(cd\left(\frac{h}{2}\right) - cd(h))\left(\frac{1}{1-0.25}\right)$

$= \mathrm{c}d\left(\frac{h}{2}\right) + \frac{1}{3}\ (cd\left(\frac{h}{2}\right) - cd(h))$

or $\mathrm{c}d\left(\frac{h}{4}\right) + \frac{1}{3}\ (cd\left(\frac{h}{4}\right) - cd(\frac{h}{2}))$ (for example)

(6) Forward difference method:

limit of $fd \approx fd\left(\frac{h}{2}\right) + 0.5(fd\left(\frac{h}{2}\right) - fd(h))\left(\frac{1}{1-0.5}\right)$

$= fd\left(\frac{h}{2}\right) + \ (fd\left(\frac{h}{2}\right) - fd(h))$

or $fd\left(\frac{h}{4}\right) + (fd\left(\frac{h}{4}\right) - fd(\frac{h}{2}))$ (for example)