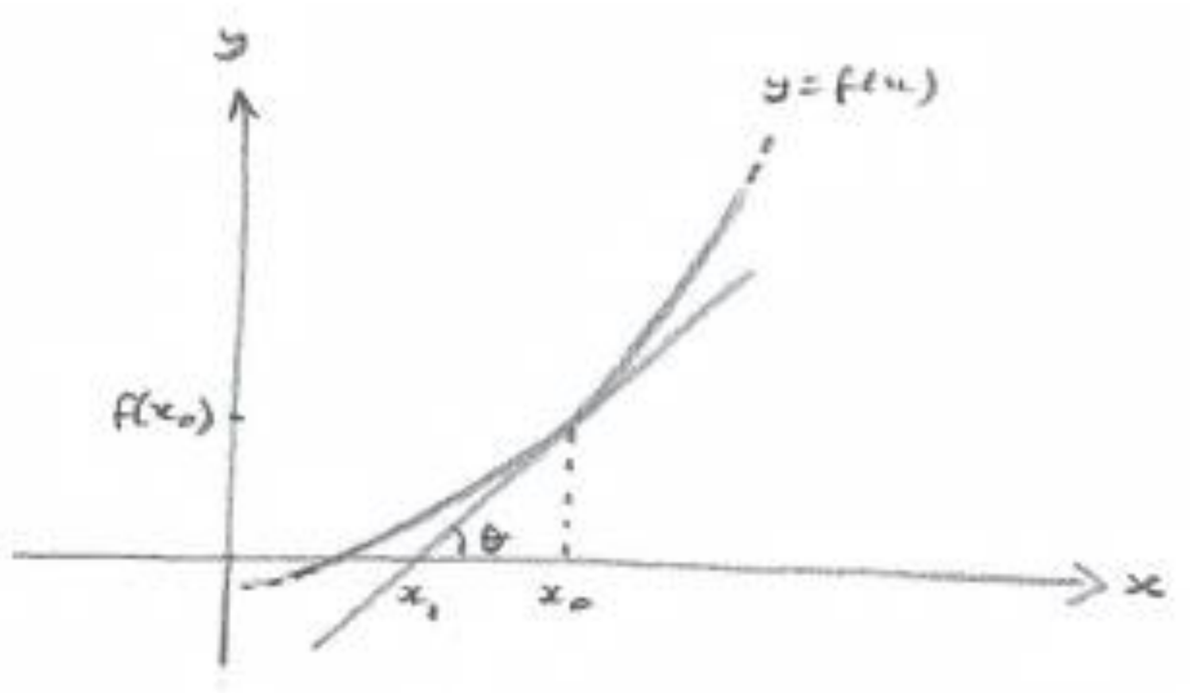
Numerical Solution of Equations - Newton-Raphson Method

(4 pages; 22/10/18)



(1) $f'(x_0) = tan\theta = \frac{f(x_0)}{x_0 - x_1}$,

so that $x_1 = x_0 - (x_0 - x_1) = x_0 - \frac{f(x_0)}{f'(x_0)}$

(2) A calculator can be used to find the root of

$x^3 - x - 1 = 0$, with $x_0 = 1.5$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $\quad f'(x) = 3x^2 - 1$

Type in the following:

$1.5 =$

$Ans - (Ans^3 - Ans - 1) \div (3Ans^2 - 1)$

$= [\text{repeatedly}]$

This produces:

$x_0 = 1.5$

$x_1 = 1.34783$

$x_2 = 1.32520$

$x_3 = 1.32472$

$x_4 = 1.32472$

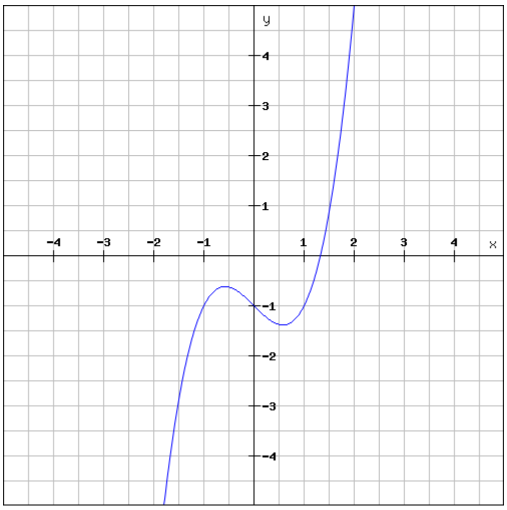
(3) The Newton-Raphson method breaks down if:

(a) $f'(x_0) = 0$; ie at a stationary point

(b) $f'(x_0)$ is close to 0 and causes x_1 to be in another region (e.g. nearer another root, or where $f(x)$ isn't defined)

Example: $f(x) = x^3 - x - 1$; $f'(x) = 3x^2 - 1$

$f'(x) = 0 \Rightarrow x = \frac{1}{\sqrt{3}} = 0.57735$ (5sf)



r	x(r)	x(r)	x(r)	x(r)
0	1.5	0.5	0.57735	0.578350269
1	1.347826	-5	6.24E+15	400.0181707
2	1.3252	-3.364864865	4.16E+15	266.6793381
3	1.324718	-2.280955054	2.77E+15	177.7870634
4	1.324718	-1.556276568	1.85E+15	118.5259694
5	1.324718	-1.043505227	1.23E+15	79.01921158
6	1.324718	-0.561409519	8.21E+14	52.68234018
7	1.324718	-11.86434492	5.48E+14	35.1258989
8	1.324718	-7.925964324	3.65E+14	23.42386433
9	1.324718	-5.306828631	2.43E+14	15.62601021
10	1.324718	-3.568284223	1.62E+14	10.43294791
11	1.324718	-2.41592421	1.08E+14	6.979735902
12	1.324718	-1.647600608	7.21E+13	4.692104232
13	1.324718	-1.112174715	4.81E+13	3.191531847
14	1.324718	-0.646071914	3.2E+13	2.23350451
15	1.324718	1.826323486	2.14E+13	1.66722651
16	1.324718	1.463768988	1.42E+13	1.399194342
17	1.324718	1.339865398	9.5E+12	1.329410891
18	1.324718	1.324927456	6.33E+12	1.324738351
19	1.324718	1.324717998	4.22E+12	1.324717958

The above table shows that the closer x_0 is to the stationary point, the longer the method takes to converge; with no convergence at all at the stationary point itself.

(4) The Newton Raphson formula can be written in the form

$x_{n+1} = g(x_n)$, where $g'(\alpha) = 0$ (and α is the relevant solution of $f(x) = 0$).

This means that the Newton Raphson method is a special case of the Fixed Point method, with 2nd order convergence (assuming that the method doesn't break down); i.e. $e_{n+1} \approx k(e_n)^2$ (where $e_n = x_n - \alpha$) [see "Fixed Point method"].

Proof: $x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Writing $g(x) = x - \frac{f(x)}{f'(x)}$,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{\left(f'(x)\right)^2}$$

Then, as $f(\alpha) = 0$, $g'(\alpha) = 1 - \frac{\left(f'(\alpha)\right)^2}{\left(f'(\alpha)\right)^2} = 0$