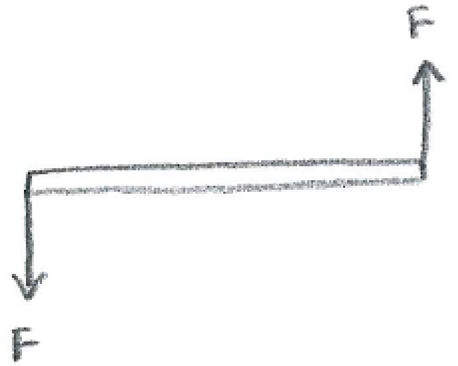
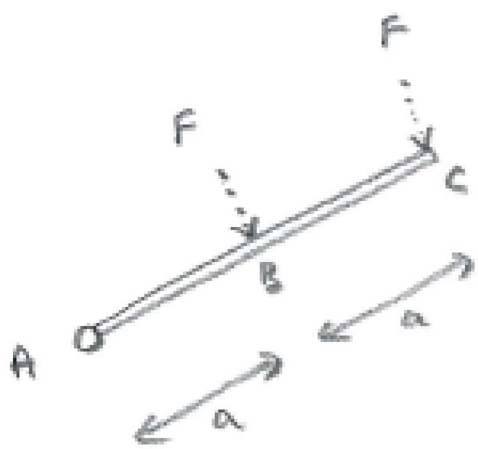
Moments (12 pages; 11/3/17)

(1) In many Mechanics models, an object is treated as a particle. This means that any rotation of the object is not considered. In some cases this isn't appropriate. For example, in the diagram below, the forces on the object balance, but clearly the object is not in 'rotational equilibrium': overall, the forces have a turning effect.



(2) In the diagram below, AC represents a door, with its hinge at A. Clearly a force F applied at C will have a greater turning effect than the same force applied at B.



The turning effect of the force at C (its moment) is defined to be

$-F(2a)$

More generally, it is the magnitude of the force × the perpendicular (ie shortest) distance of the line of action of the force from A (ie the point about which 'moments are taken'), with a negative sign if the turning is in a clockwise sense.

[Note: The vector specification of a force (ie its magnitude and direction) is not sufficient for dealing with rigid bodies: we also need to know the line along which it acts. This could be determined from a particular point where the force acts (together with the direction of the force).]

The unit of a moment of a force is the Nm.

(3) Most situations in which moments are used (at A level) concern rigid bodies 'in equilibrium'. This can usually be taken to mean that the object is stationary (though note that forces are said to be in equilibrium when an object is moving with constant velocity).

We can then resolve forces in two perpendicular directions, and use Newton's 2nd law to set up two equations

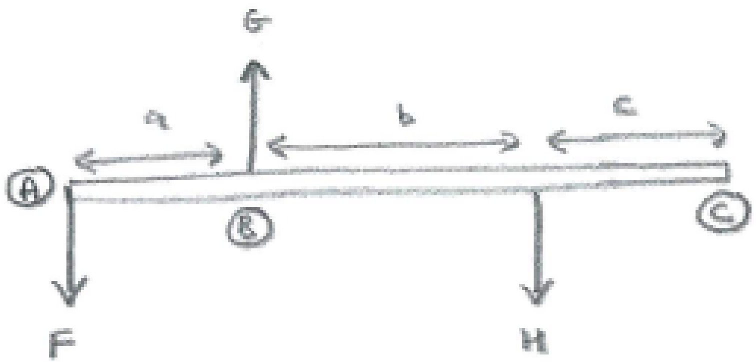
(ie $total\ force = mass\ \times accel. = 0$)

We can also use the fact that there is rotational equilibrium to say that the net moment (of all the forces on the object) is zero; ie there is no net turning effect.

The question remains as to which point to take moments about.

It will be shown next that, provided the forces are in equilibrum, it doesn't matter which point we choose. Also, the point needn't actually be within the object itself (though it usually is).

(4) Consider the rod in the diagram below, subject to the forces F, G & H. If the rod is in equilibrium, then $F + H = G$ (ie there is vertical equilibrium). [In other situations, we can also employ horizontal equilibrium.]



The net moment can be calculated about A, B or C, as follows:

Moments about A [sometimes indicated by: $M(A)$]:

$Ga - H(a + b) = (F + H)a - H(a + b) = Fa - Hb$

Moments about B:

$Fa - Hb$

Moments about C:

$F(a + b + c) - G(b + c) + Hc$

$= F(a + b + c) - (F + H)(b + c) + Hc$

$= Fa - Hb$

And, in general, when the forces are balanced, taking moments about any point will give the same result.

Then, because the rod is in rotational equilibrium, $Fa - Hb = 0$

(5) Some points will be more convenient to take moments about, for the following reasons:

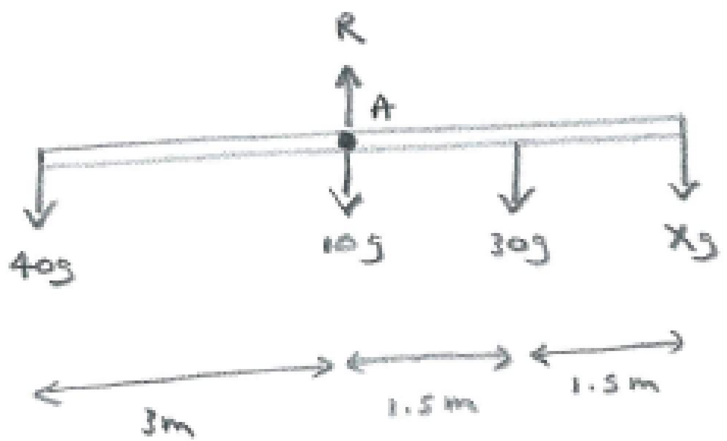
(i) If we are not interested in a particular force, or if it isn't known, then it may be avoided by taking moments about a point through which the force in question acts (eg taking moments about B, in the above example, if G is unknown; note that, in this case, the equation $F + H = G$ would not be used - in order to keep G out of the working).

(ii) Some points involve more complicated equations; eg taking moments about C in the above example. In general, take moments about a point where as many forces as possible act.

(6) Example: Children sitting on a seesaw

The children have masses $40, 30$ & $X\ kg$, and the seesaw (assumed to be uniform) has mass $10kg$.

The problem is to find X, given that the seesaw is in equilibrium.



The diagram is a force diagram for the seesaw (ie it shows all of the forces acting on the seesaw). R is the reaction force of the supporting structure on the seesaw (and is unknown).

Thus, in this case there is no need to resolve forces vertically (giving an equation involving R).

Instead, we can just take moments about A:

Rotational equilibrum $\Rightarrow$

$40g(3) - 30g(1.5) - Xg(3) = 0$,

so that $X = \frac{120-45}{3} = 25$

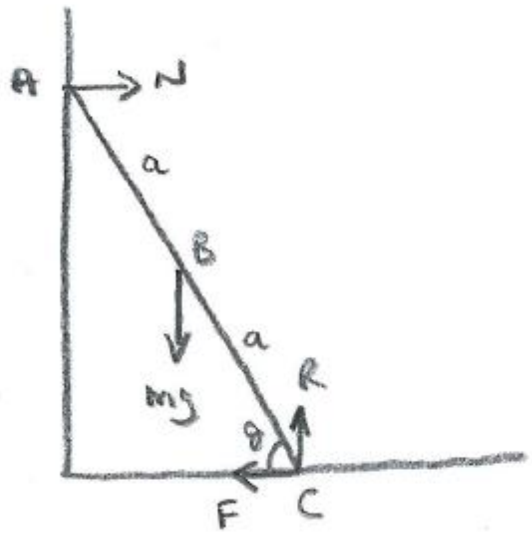
[R can then be found from $R = 40g + 10g + 30g + Xg$, if required.]

Note: As an alternative to equating the net moment to zero, we could say that the total clockwise moment equals the total anti-clockwise moment.

(7) Moments of forces at an angle

Example (ladder resting against a wall)

The ladder is of length $2a$ and mass m, and is assumed to be uniform, so that its centre of mass is at its mid-point. The wall is assumed to be smooth, so that the reaction at the wall, N is perpendicular to the wall. The coefficient of friction between the ladder and the ground is μ. Given that the ladder is resting at an angle θ to the ground, find the minimum possible value of μ in terms of θ.



Approach 1

Resolving vertically, $R = mg$.

If we take moments about A, then one approach is to extend the lines of action of the forces, in order to find the perpendicular (ie shortest) distance between those lines and A.

Thus, the perpendicular distance between mg (extended) and A is $acos\theta$; between R (extended) and A: $2acos\theta$, and between F (extended) and A: $2asin\theta$.

Then rotational equilibrium ⇒ the net moment about A is zero,

so that $-mg(acos\theta) + R(2acos\theta) - F(2asin\theta) = 0$

Also, in the limiting case, where the ladder is about to slip,

$F = \mu R = \mu mg$

Thus $-cos\theta + 2cos\theta - 2\mu sin\theta = 0$,

so that $\mu = \frac{cos\theta}{2sin\theta} = \frac{1}{2}cot\theta$

(Note that, for larger θ, a smaller value of μ will be sufficient to keep the ladder in place.)

Approach 2

An alternative way of finding the moments of the forces is to resolve each force, at a suitable point on its line of action, in two convenient perpendicular directions.

Thus, mg can be resolved at B into components along and perpendicular to the ladder. The component along the ladder then has no moment about A, whilst the component perpendicular to the ladder ($mgcos\theta$) has moment $-(mgcos\theta)a$.

Similarly, R has moment $(Rcos\theta)(2a)$, whilst F has moment

$-(Fsin\theta)(2a)$.

Thus each of the moment terms is the same as before.

Note: In the case of the mg force, B is the best place to resolve the force (as one of the components has zero moment about A). However, it can be shown that the same total moment would be obtained if the force were resolved at some other point on its line of action.

(8) Alternative approaches

Once forces have been resolved in two perpendicular directions and moments taken about a particular point, so that 3 equations have been created, it isn't possible to obtain an independent 4th equation by taking moments about another point; ie it will just duplicate information already obtained.

However, it is possible to take moments about 2 points and resolve forces in just one direction - provided that this direction isn't perpendicular to the line joining the 2 points.

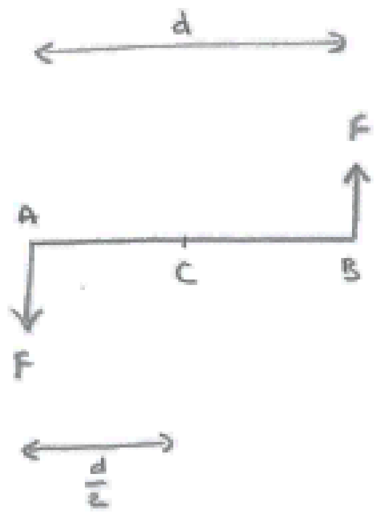
Alternatively, it is possible to take moments about 3 points (and do no resolving of forces) - provided that the 3 points don't lie on a straight line.

As it is usually simpler to resolve forces, rather than take moments, these alternative methods are not normally used. They could be used as a check though.

(9) Couples

The term 'couple' is used to describe a pair of equal but opposite forces, applied to an object, which don't have the same line of action, so that there is a turning effect. (It is sometimes also used in the more general situation where there are more than two forces, which have a resultant of zero but a net non-zero moment.)

As before, the fact that the forces are balanced means that it doesn't matter which point we take moments about. Thus, referring to the diagram below, we could take moments about A, for example, to give a net moment of Fd.



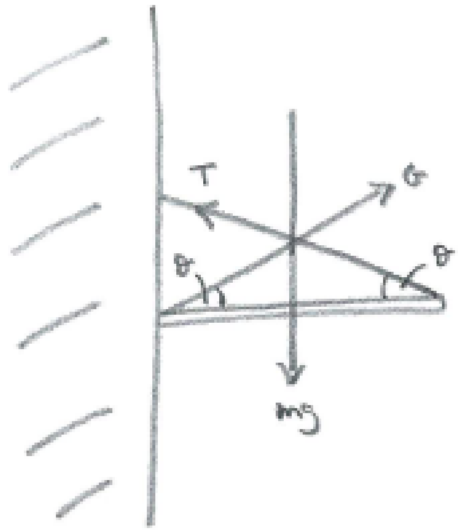
(10) Hinged Joints

Suppose that a rod is attached to a wall by a hinged joint (ie so that the angle can be varied). The hinge will often be described as 'smooth'. This means that it offers no resistance to being turned; ie there is no moment within the hinge countering any external forces. (Were the hinge not to be smooth then the resistance to turning within the hinge could be thought of as due to the moment of a frictional force acting at a short distance from the centre of the hinge.)

(11) Reaction forces at a surface

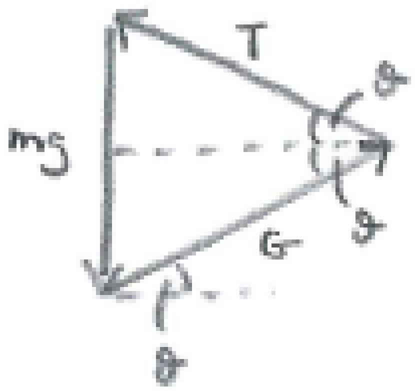
If a rod, say, is attached to a surface, then there will be a reaction force on the rod, at a particular angle. In practice, it is usually convenient to resolve this reaction force into two perpendicular components: along and perpendicular to the surface. Were the rod to be resting on the surface (say, if it were a ladder), then the component along the surface would be the frictional force.

(12) Alternative approach to equilibrium problem



Normally we show the reaction at the wall as being made up of two perpendicular components. However, if there are only two other forces, in addition to the reaction at the wall R, then we can take advantage of the fact that the 3 forces must be concurrent (ie their lines of action must meet at a point), in order for there to be rotational equilibrium. (Were the forces not to meet at a point, then taking moments about the point of intersection of two of the forces would give a non-zero moment, which would mean that the object was not in rotational equilibrium.)

Because the line of action of the force mg bisects the rod, there are two congruent right-angled triangles in the diagram, with the same corresponding angle θ.



Then, as the forces are balanced, they must form a vector triangle, as shown in the 2nd diagram.

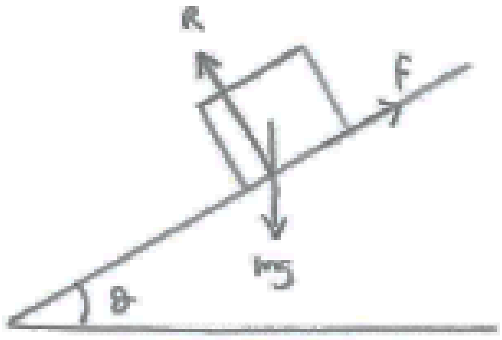
Hence, from the lower right-angled triangle, $\frac{\left(\frac{mg}{2}\right)}{G} = sin\theta$,

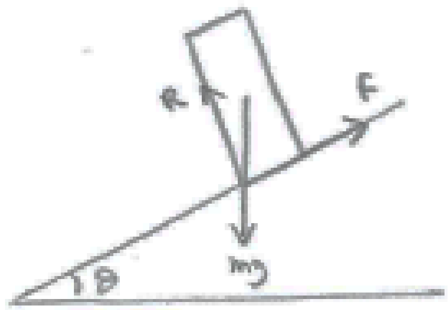
so that the horizontal and vertical components of G are

$Gcos\theta = \left(\frac{mg}{2}\right)\frac{cos\theta}{sin\theta} = \frac{mg}{2tan\theta}$

and $Gsin\theta = \frac{mg}{2}$

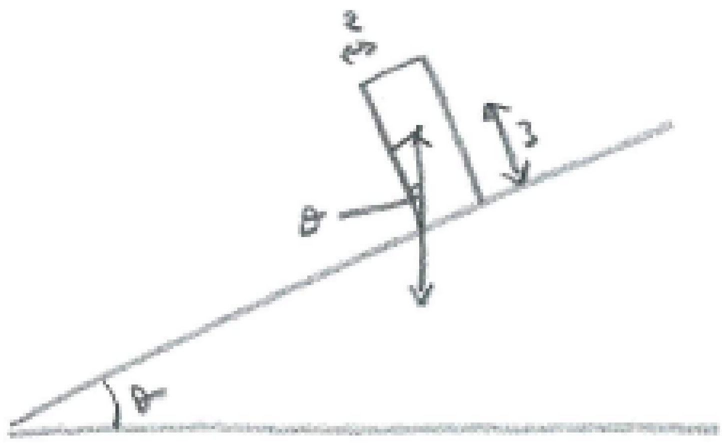
(13) Toppling on a slope



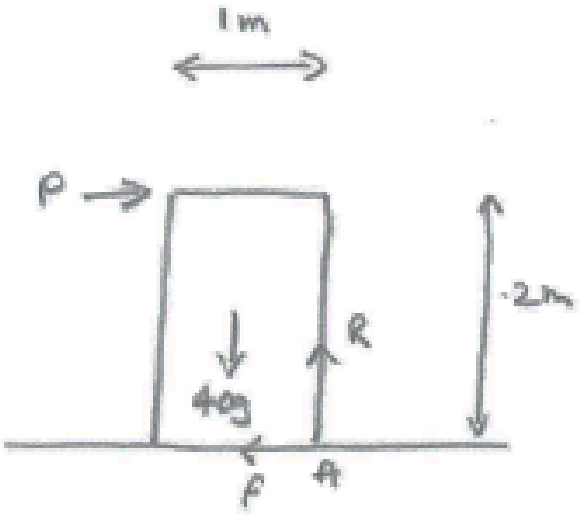


As discussed above, whenever there are 3 forces on an object that is in rotational equilibrium, the lines of action of these forces must meet at a common point. In the left hand diagram (where F is the frictional force), the normal reaction R must be acting at the point shown, in order for the forces to meet at a common point. In the right hand diagram, the extreme situation is reached where R acts at the left hand corner (about which the object is liable to topple). Any increase in θ would cause toppling to occur, as it would not be possible for the line of action of R to be any further to the left.

Thus the angle of toppling is determined by the centre of mass of the block being directly above the left hand corner. Thus, in the diagram below, $tan\theta = \frac{2}{3}$ (Note, as a check on the angle, that when the block is on a level surface, it points directly upwards and its left hand edge makes an angle of 0° with the vertical.)



(14) Toppling by an applied force



When the block is about to topple, it will be pivoting about A, so that R must act at A (as this is the only point of contact between the block and the surface).

As the block is not yet rotating, the total moment about A must be zero.

Thus $-2P + 40g(0.5) = 0$,

so that $P = \frac{1}{2}(40)(9.8)(0.5) = 98\ N$