Method of Differences (2 pages; 28/1/19)

(1) Example: $\sum_{r=1}^{n}\{\frac{1}{2(r+1)}-\frac{1}{(r+2)}+\frac{1}{2(r+3)}\}$

$=\frac{1}{2}\sum_{r=1}^{n}\{\frac{1}{(r+1)}-\frac{2}{(r+2)}+\frac{1}{(r+3)}\}$

$=\frac{1}{2}S$,

$$\begin{aligned}\text{where } S = &\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right)\\ &-2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\right)\\ &+\left(\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)\end{aligned}$$

Highlighted items cancel,

so that required sum $=\frac{1}{2}\left\{\frac{5}{6}-\frac{2}{3}-\frac{1}{n+2}+\frac{1}{n+3}\right\}$

$=\frac{1}{12}-\frac{1}{2(n+2)}+\frac{1}{2(n+3)}$

Notes

(i) A handwritten version would have a box round the highlighted items.

(ii) For exam purposes, a typical requirement would be to see at least two items cancelling at the start (where the first item is $\frac{1}{4}-\frac{2}{4}+\frac{1}{4}$) , and perhaps one at the end (though two would be best, to be on the safe side). Leave plenty of space for extra terms to be inserted.

(iii) This layout is usually clearer than:

$\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{2}{6}+\frac{1}{7}\right)+\cdots$

(iv) In this case, it probably isn't worth simplifying to a single fraction (unless asked to do so).

Had the answer been $\frac{1}{2(n+2)} - \frac{1}{2(n+3)}$, then

the form $\frac{(n+3)-(n+2)}{2(n+2)(n+3)} = \frac{1}{2(n+2)(n+3)}$ would usually be preferable.

(v) A check can be made by substituting $n = 1$:

$\frac{1}{2(1+1)} - \frac{1}{(1+2)} + \frac{1}{2(1+3)} = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24}$

and $\frac{1}{12} - \frac{1}{2(1+2)} + \frac{1}{2(1+3)} = \frac{1}{12} - \frac{1}{6} + \frac{1}{8} = \frac{2-4+3}{24} = \frac{1}{24}$ also

(2) If asked to show that $\frac{1-r}{(r+1)(r+2)} = \frac{2}{r+1} - \frac{3}{r+2}$, for example, then there is no need to use Partial Fractions: instead, just rearrange the RHS to obtain the LHS.

In this connection (for a different example), beware of writing $\frac{1}{r+1} - \frac{1}{r+2} = \frac{r+2-r+1}{(r+1)(r+2)}$ when you mean $\frac{r+2-(r+1)}{(r+1)(r+2)}$

(you may lose a mark - especially as it's a 'show that' result; ie $\frac{r+2-r+1}{(r+1)(r+2)} = \frac{1}{(r+1)(r+2)}$ won't be sufficient).