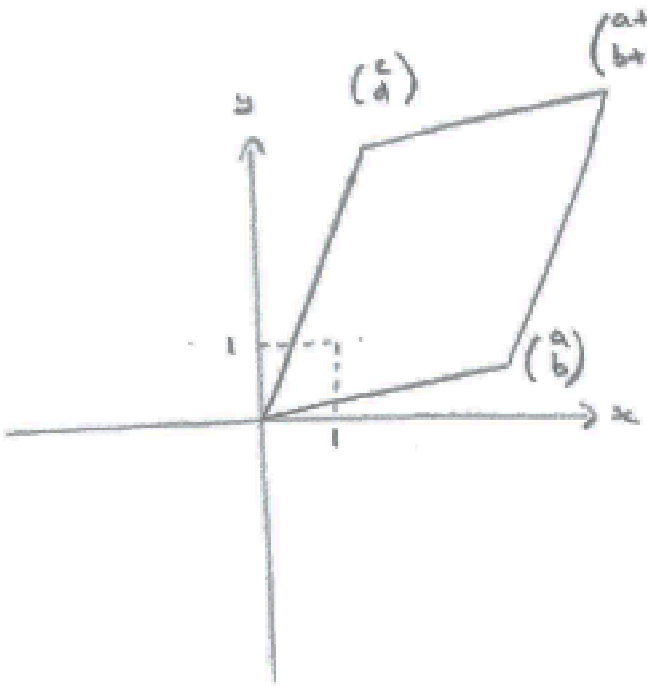
Matrix Transformations (9 pages; 1/10/19)

(1) General

(i) The transformation represented by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} c \\ d \end{pmatrix}$.

(ii) In general, the image of the unit square is a parallelogram (see diagram).

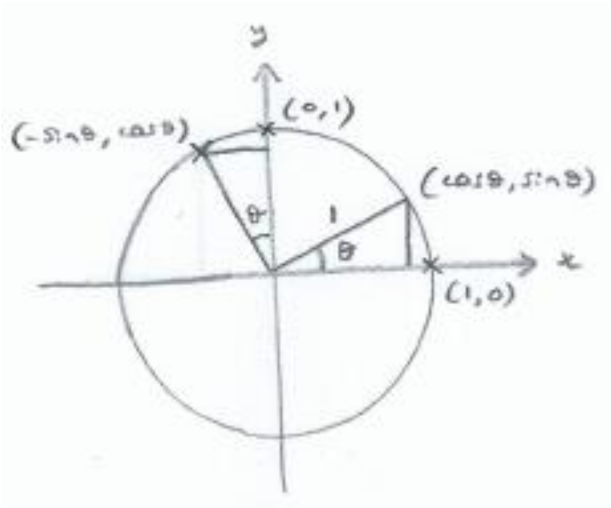


(iii) The area scale factor of the transformation is the determinant of the matrix, as can be seen by finding the area of the above parallelogram (see Matrices - Exercises (Part 1)).

Thus, for pure rotations (ie not involving any stretching), the determinant will be 1. For pure reflections, it will be -1, due to the reversal of the order of points on the edge of any shape being reflected.

(2) Rotations

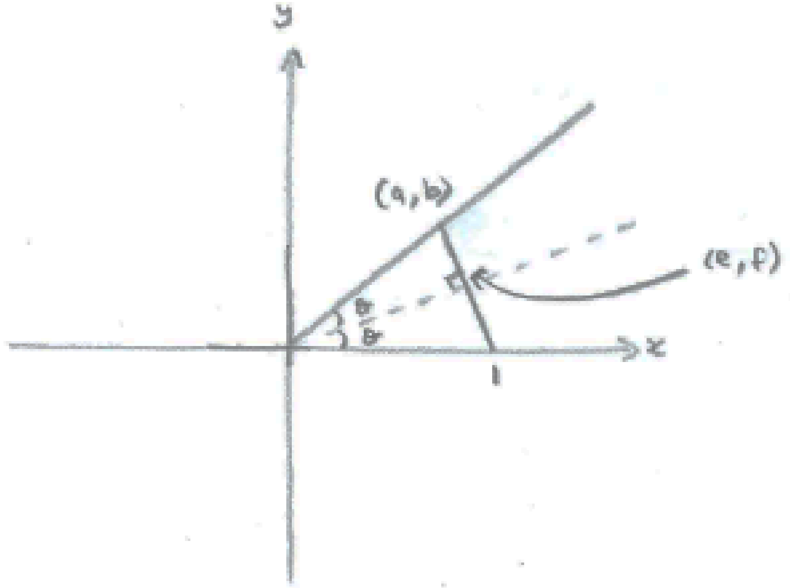
To show that a rotation of θ (anti-clockwise) is represented by the matrix $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$:



Referring to the diagram above, if the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is rotated through an angle of θ (anti-clockwise), then its image will be $\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$.
Similarly, the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be $\begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix}$.

(3) Reflections

The matrix for reflection in the line $y = tan\theta . x$ can be found by considering the images of the points (1,0) and (e,f) (see diagram).



Let the required matrix be $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Using column vectors:

The image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} cos2\theta \\ sin2\theta \end{pmatrix}$

As $\begin{pmatrix} e \\ f \end{pmatrix}$ lies on the mirror line,

$\begin{pmatrix} cos2\theta & c \\ sin2\theta & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

$\Rightarrow \begin{pmatrix} cos2\theta & c \\ sin2\theta & d \end{pmatrix}\begin{pmatrix} cos\theta cos\theta \\ cos\theta sin\theta \end{pmatrix} = \begin{pmatrix} cos\theta cos\theta \\ cos\theta sin\theta \end{pmatrix}$

[since the distance of $\begin{pmatrix} e \\ f \end{pmatrix}$ from the origin is $cos\theta$]

$\Rightarrow cos2\theta cos^2\theta + c.cos\theta sin\theta = cos^2\theta$

& $sin2\theta cos^2\theta + dcos\theta sin\theta = cos\theta sin\theta$

$\Rightarrow c = \frac{cos^2\theta(1-cos2\theta)}{cos\theta sin\theta} = \frac{cos\theta.2sin^2\theta}{sin\theta}$

$\Rightarrow c = 2sin\theta cos\theta = sin2\theta$

and $d = 1 - 2cos^2\theta = sin^2\theta - cos^2\theta = -cos2\theta$

Thus $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$

[Note that $\begin{vmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{vmatrix} = -1$, as you would expect for a reflection.]

[See separate note for Shears.]

(4) Zero determinant

(4.1) Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

where $ad - bc = 0$, so that $ad = bc$ & $\frac{d}{c} = \frac{b}{a} = k$, say

ie $b = ka$ & $d = kc$

So $ap + cq = u$ (1)

and $bp + dq = v \Rightarrow kap + kcq = v \Rightarrow k(ap + cq) = v$ (2)

Then (1) & (2) $\Rightarrow v = ku$ (a straight line through the origin)

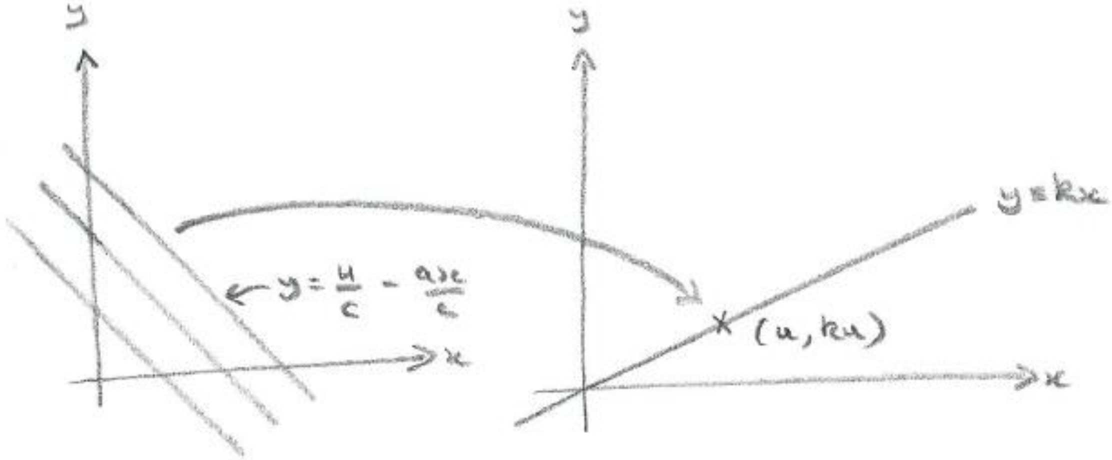
Thus, the gradient of the single image line is $\boldsymbol{\frac{b}{a}}$.

(4.2) Also, for a given point (u, ku), (1) $\Rightarrow cq = u - ap$

$\Rightarrow q = \frac{u}{c} - \frac{ap}{c}$; ie the possible points (p, q) lie on a straight line

(see diagram below)

Thus, the lines mapping to a specific point on the image line all have gradient $\boldsymbol{-\frac{a}{c}}$.



(4.3) Exercise: Show that, if $\begin{pmatrix} p \\ q \end{pmatrix}$ is a point on the image line, then the image of $\begin{pmatrix} p \\ q \end{pmatrix}$ is $(a+d)\begin{pmatrix} p \\ q \end{pmatrix}$

Solution

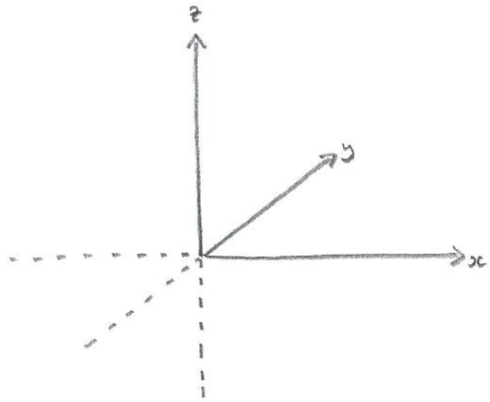
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ \frac{bp}{a} \end{pmatrix} = \begin{pmatrix} ap + \frac{bcp}{a} \\ bp + \frac{bdp}{a} \end{pmatrix},$$

and $ap + \frac{bcp}{a} = ap + dp$, as $\frac{d}{c} = \frac{b}{a}$;

ie $(a+d)p$

and, as the image of $\begin{pmatrix} p \\ q \end{pmatrix}$ is known to lie on $y = \frac{b}{a}x$, it follows that the y-coord. of the image is $(a+d)p\left(\frac{b}{a}\right) = (a+d)p\left(\frac{q}{p}\right) = (a+d)q$, as required.

(5) 3×3 Transformations



(i) The matrix $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$ maps $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ onto $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ onto $\begin{pmatrix} g \\ h \\ i \end{pmatrix}$

In general (if $\det \neq 0$), a cube is transformed to a parallelepiped.

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & e & h \\ 0 & f & i \end{pmatrix}$ maps $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ onto $\begin{pmatrix} p \\ q' \\ r' \end{pmatrix}$

ie the x-coordinate remains the same, so that the transformation of a point takes place in a plane parallel to the plane $x = 0$ (ie the y-z plane), and $\begin{pmatrix} e & h \\ f & i \end{pmatrix}$ determines the transformation in that plane

Similarly, $\begin{pmatrix} a & 0 & g \\ 0 & 1 & 0 \\ c & 0 & i \end{pmatrix}$ maps $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ onto $\begin{pmatrix} p' \\ q \\ r' \end{pmatrix}$

and $\begin{pmatrix} a & d & 0 \\ b & e & 0 \\ 0 & 0 & 1 \end{pmatrix}$ maps $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ onto $\begin{pmatrix} p' \\ q' \\ r \end{pmatrix}$

Example: $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ represents a 90° rotation about the z-axis

Note: The expression "clockwise" isn't used in 3D (but in the diagram above, a positive rotation about the z-axis would be anti-clockwise if projected onto the x-y plane).

Exercise: Describe the transformation represented by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Solution

y coordinate stays the same

$\therefore$ transformation in a plane parallel to the $x - z$ plane

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection about the line $y = x$

So $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ represents a reflection about the plane $\text{z} = x$

(iii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ represents a reflection in the plane $z = 0$ (aka the x-y plane), as the x & y coordinates are unchanged;

also $\det = -1$ suggests reflection

(iv) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ represents an enlargement of scale factor 2,

centre the origin

(v) $\begin{pmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{pmatrix}$ represents a $\theta°$ rotation about the y-axis

(vi) **Exercise**: Find the effect of a reflection in the plane $y = 0$,

followed by a reflection in the plane $x = 0$

Solution

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$=$ 180° rotation about z-axis

(vii) Example: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a+c \\ b \\ 0 \end{pmatrix}$

All points map to the plane $z = 0$; ie the $x - y$ plane (note that the determinant is zero).

(viii) Effect of a transformation on a line

eg effect of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ on $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

[If the line is expressed in Cartesian form, convert to above parametric form.]

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$$

So, in general, a line is transformed to another line.

(ix) Effect of a transformation on a plane

eg effect of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$ on $2x + y - z = 1$

Step 1: Convert the plane into parametric form

Let $x = s$ and $y = t$, so that a general point is $\begin{pmatrix} s \\ t \\ 2s + t - 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Step 2

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mathrm{t}\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Step 3: Convert back into Cartesian form (if required)

[See "Vector Theory".]

So, in general, a plane is transformed to another plane.