Matrices - Exercises: Transformations (Solutions)

(14 pages; 13/8/19)

(1) How can you tell if a matrix represents a (pure) reflection / rotation?

Solution

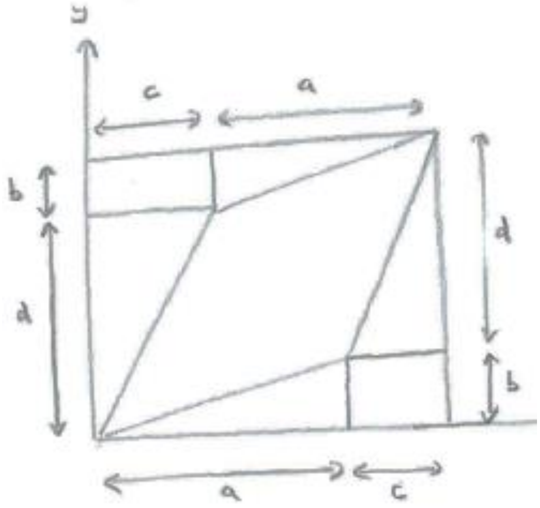
Reflection if $\det = -1$; rotation if $\det = 1$

(2) Derive a formula for the area of a triangle with corners at $(0,0), (a,b), (c,d)$, using matrix transformations.

Solution

The formula for the area of a triangle with corners $(0,0), (a,b), (c,d)$ can be obtained by considering the matrix transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$: (a,b) is the image of $(1,0)$ and (c,d) is the image of $(0,1)$; the area of the triangle with corners $(0,0), (1,0), (0,1)$ is $1/2$, and the area scale factor is $|ad-bc|$, since $ad-bc$ is the determinant of the matrix (the modulus sign only being needed when the order of the corners becomes reversed in the course of the transformation).

(3) Use the diagram below to show that the area scale factor of the transformation represented by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the determinant of the matrix.



Solution

Area of parallelogram $= (a+c)(b+d) - 2bc - 2.\frac{1}{2}ab - 2.\frac{1}{2}cd$

$= ab + ad + bc + cd - 2bc - ab - cd = ad - bc$

(4) Find the equation of the line that the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ maps all points to, and find the equation of the line that maps to the point (1,3)

Solution

$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow p + 2q = u$

and $3p + 6q = v$

so that $v = 3u$; equation of line is $y = 3x$

$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow p + 2q = 1$

$\Rightarrow q = \frac{1}{2}(1 - p)$

ie equation is $y = \frac{1}{2} - \frac{x}{2}$

(5) Find the equations of the invariant lines of the transformation represented by the matrix $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$

Solution

[Note that the transformation is a shear, as the determinant is 1, and the trace $(4 + (-2))$ equals 2. Note also that, in general, the eigenvectors of a matrix give the invariant lines that pass through the Origin. In the case of a shear, which has repeated eigenvalues of 1 (see "Matrices - Notes"), the single eigenvector is the line of invariant points, and the other invariant lines will be parallel to this.]

Suppose that $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ for all x

Then $4x + 3(mx + c) = x'$

and $-3x - 2(mx + c) = mx' + c$

$\Rightarrow -3x - 2mx - 2c = m(4x + 3mx + 3c) + c$

$\Rightarrow x(-3 - 2m - 4m - 3m^2) - 2c - 3mc - c = 0$

$\Rightarrow x(3m^2 + 6m + 3) + 3c + 3mc = 0$

$\Rightarrow x(m^2 + 2m + 1) + c(1 + m) = 0$

Equating coeffs of powers of x,

$m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1$

and either $c = 0$ *or* $m = -1$

Thus $m = -1$ and c can take any value,

so that the invariant lines are $y = -x + c$

(6) Prove that invariant points of a 2D transformation always lie on a line passing through the Origin.

Solution

Suppose that $y = px + q$ is a line of invariant points for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ px+q \end{pmatrix} = \begin{pmatrix} x \\ px+q \end{pmatrix}$

$\Rightarrow ax + cpx + cq = x$

and $bx + dpx + dq = px + q$

As these equations are to hold for any x, we can equate coefficients of x and the constant terms to give:

$a + cp = 1$ (1)

$cq = 0$ (2)

$b + dp = p$ (3)

$dq = q$ (4)

Suppose that $q \neq 0$

Then (2) & (4) $\Rightarrow c = 0$ & $d = 1$

and (1) & (3) $\Rightarrow a = 1$ & $b = 0$

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in which case all points are invariant.

Thus, apart from this trivial case, $q = 0$ and so a line of invariant points takes the form $y = px$, and therefore passes through the Origin.

(7) In 3 dimensions, find the effect of a reflection in the plane $y = 0$, followed by a reflection in the plane $x = 0$

Solution

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is a

180° rotation about the z-axis

(8) Show that the matrix $\begin{pmatrix} cos2\theta & sin2\theta \\ sin2\theta & -cos2\theta \end{pmatrix}$ [representing a reflection in the line $y = tan\theta.x$] can be written as $\begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}$

- where $m = tan\theta$

Solution

$tan2\theta = \frac{2tan\theta}{1-tan^2\theta} = \frac{2m}{1-m^2}$

The right-angled triangle with opposite and adjacent sides of $2m$ & $1 - m^2$ has a hypotenuse of $\sqrt{4m^2 + (1 - 2m^2 + m^4)}$

$= \sqrt{(1 + m^2)^2} = 1 + m^2$,

so that $sin2\theta = \frac{2m}{1+m^2}$ & $cos2\theta = \frac{1-m^2}{1+m^2}$, as required

(9) For the transformation matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$, where a, b, c & d are positive, find a relationship between the trace $a + d$ and the determinant that must hold in order for the transformation to have an invariant line that doesn't pass through the Origin.

Solution

Suppose that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} x' \\ mx'+k \end{pmatrix}$ for all x,

where $k \neq 0$

Then $ax + cmx + ck = x'$ & $bx + dmx + dk = mx' + k$

and $(a + cm)x + ck = x'$ & $(b + dm)x + (d - 1)k = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(a + cm)x + mck = (b + dm)x + (d - 1)k$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(a + cm) = b + dm$ & $mck = (d - 1)k$ (1)

Thus, as k & $c \neq 0$,

$cm^2 + (a - d)m - b = 0$ & $m = \frac{d-1}{c}$,

and hence $c\left(\frac{d-1}{c}\right)^2 + (a - d)\left(\frac{d-1}{c}\right) - b = 0$,

so that $(d - 1)^2 + (a - d)(d - 1) - bc = 0$

and $(d - 1)(d - 1 + a - d) - bc = 0$,

giving $(d - 1)(a - 1) - bc = 0$

and hence $ad - dc - (a + d) + 1 = 0$

or $det = trace - 1$

(10) For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

Solution

[We can find the invariant lines that pass through the Origin from the eigenvectors, but the result of Exercise (6) indicates that there will also be an invariant line that doesn't pass through the Origin.]

Suppose that $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ for all x,

Then $2x + 2mx + 2c = x'$ & $x + 3mx + 3c = mx' + c$

and $(2 + 2m)x + 2c = x'$ & $(1 + 3m)x + 2c = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(2 + 2m)x + 2mc = (1 + 3m)x + 2c$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(2 + 2m) = 1 + 3m$ & $2mc = 2c$ (1)

Case 1: $c = 0$ (invariant lines through the Origin; ie the eigenvectors)

$2m^2 - m - 1 = 0 \Rightarrow (2m + 1)(m - 1) = 0$,

so that $m = -\frac{1}{2}$ & 1

[Check: To find the eigenvalues of the matrix, we create the characteristic equation $\begin{vmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(3 - \lambda) - 2 = 0$

and $\lambda^2 - 5\lambda + 4 = 0$, so that $\lambda_1 = 1$ & $\lambda_2 = 4$

Then, to find the eigenvectors:

$\begin{pmatrix} 2 - 1 & 2 \\ 1 & 3 - 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that $x + 2y = 0$; ie one invariant line is $y = -\frac{1}{2}x$

and $\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $x - y = 0$; ie another invariant line is $y = x$

(and these agree with the values of m above)]

Case 2: $c \neq 0$

From (1), $m = 1$ (satisfying both of the equations, in agreement with Exercise (6))

Thus, the other invariant lines are those of the form $y = x + c$

[Also, from Exercise (6), we expect $m = \frac{d-1}{c} = \frac{3-1}{2} = 1$]

(11) If the condition found in Exercise (8) applies, what can be deduced about the eigenvalues of the transformation?

Solution

As $\lambda_1 + \lambda_2 = a + d$ & $\lambda_1\lambda_2 = ad - bc$,

$ad - bc - (a + d) + 1 = 0 \Rightarrow \lambda_1\lambda_2 - (\lambda_1 + \lambda_2) + 1 = 0$,

so that $\lambda_2(\lambda_1 - 1) = \lambda_1 - 1$

$\Rightarrow$ either $\lambda_1 = 1$ or $\lambda_2 = 1$

This means that at least one of the invariant lines through the Origin will also be a line of invariant points.

(12) For a reflection in the line $y = x$:

(i) Find the transformation matrix.

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) By drawing a diagram, are there any invariant lines that don't pass through the Origin?

(iv) Are there any lines of invariant points?

Solution

(i) $\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}0\\1\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$, so matrix is $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

(ii) Characteristic equation is $\begin{vmatrix}0-\lambda & 1\\1 & 0-\lambda\end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

[Check: trace of $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = 0 + 0 = \lambda_1 + \lambda_2$ & $\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix} = -1 = \lambda_1\lambda_2$]

For $\boldsymbol{\lambda = 1}$:

$$\begin{pmatrix}0-1 & 1\\1 & 0-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

so that $-x + y = 0$, giving an eigenvector of $\begin{pmatrix}1\\1\end{pmatrix}$

For $\boldsymbol{\lambda = -1}$:

$$\begin{pmatrix}0+1 & 1\\1 & 0+1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

so that $x + y = 0$, giving an eigenvector of $\begin{pmatrix}1\\-1\end{pmatrix}$

Thus the invariant lines through the Origin are $y = x$ & $y = -x$

(iii) Any line parallel to $y = -x$ is an invariant line. So any line of the form $y = -x + c$, where $c \neq 0$, is an invariant line that doesn't pass through the Origin.

(iv) Invariant points satisfy $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$ and are therefore multiples of the eigenvector with eigenvalue 1.

So $y = x$ is the (only) line of invariant points.

(13) Consider the transformation represented by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(i) What type of transformation is this?

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) What can be said about the line $y = 0$?

(iv) What can be said about the line $y = c$? , where $c \neq 0$

Solution

(i) A shear in the x-direction [as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is unchanged and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ moves parallel to the x-axis]; the image of a point not on the x-axis would also need to be specified, in order to define the shear fully.

(ii)&(iii) $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$

To find the associated eigenvector: $\begin{pmatrix} 1-1 & 2 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $2y = 0$; ie the eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which means that $y = 0$ is an invariant line, and also a line of invariant points (as $\lambda = 1$)

(iv) As the transformation is a shear, lines parallel to the line of invariant points will be invariant lines.

Note: General invariant lines can be obtained by finding solutions to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} x' \\ mx'+c \end{pmatrix}$

(14) Consider the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$

(i) Write down the eigenvalues, without forming the characteristic equation.

(ii) Find the invariant lines passing through the Origin.

(iii) Show that all points in the plane are transformed to points on a specific line (to be found).

(iv) Find the line whose points are transformed to (1,3).

(v) State the line whose points are transformed to the Origin.

Solution

(i) $\lambda_1 + \lambda_2 = 1 + 6 = 7$ & $\lambda_1\lambda_2 = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$

Hence $\lambda_1 = 0$ & $\lambda_2 = 7$

(ii) **For $\boldsymbol{\lambda} = \mathbf{0}$**:

$$\begin{pmatrix} 1-0 & 2 \\ 3 & 6-0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $x + 2y = 0$, giving an eigenvector of $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

For $\boldsymbol{\lambda} = \mathbf{7}$:

$$\begin{pmatrix} 1-7 & 2 \\ 3 & 6-7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $-6x + 2y = 0$, giving an eigenvector of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Thus the invariant lines through the Origin are

$y = -\frac{1}{2}x$ & $y = 3x$

(iii) [As $\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$, the area scale factor of the transformation is 0, which means that all shapes collapse to points on a particular line]

$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow p + 2q = u$

and $3p + 6q = v$

so that $v = 3u$; equation of line is $y = 3x$ (ie one of the eigenvectors, and invariant lines)

(iv) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow p + 2q = 1$

$\Rightarrow q = \frac{1}{2}(1 - p)$

ie equation is $y = \frac{1}{2} - \frac{x}{2}$

(v) The other eigenvector, $y = -\frac{1}{2}x$ has an eigenvalue of 0, and so all its points are transformed to the Origin $[\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow 0 \begin{pmatrix} x \\ y \end{pmatrix}]$

(15) Find the invariant points and lines for the transformation represented by

the matrix $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$.

Solution

Invariant points satisfy $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 5x + 4y = x, -4x - 3y = y \Rightarrow y = -x$

This is a line of invariant points.

For points on the line $y = mx + c$,

$\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} 5x+4mx+4c \\ -4x-3mx-3c \end{pmatrix}$

For this to be an invariant line,

$-4x - 3mx - 3c = m(5x + 4mx + 4c) + c$ for all x

Equating coefficients of x, $-4 - 3m = 5m + 4m^2$,

so that $4m^2 + 8m + 4 = 0$, or $m^2 + 2m + 1 = 0$;

ie $(m+1)^2 = 0$, so that $m = -1$

Equating the constant terms, $-3c = 4mc + c$

$\Rightarrow c(4m + 4) = 0$

$\Rightarrow c = 0$ or $m = -1$

So the overall condition is: $m = -1$ and c can take any value,

and the invariant lines are of the form $y = c - x$ (including the line of invariant points $y = -x$).

[Note: $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$ represents a shear, as its determinant is 1 and the sum of the elements on the leading diagonal is 2. It follows that the invariant lines will all be parallel to the line of invariant points.]

(16)(i) Show that the transformation represented by the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ (with determinant zero) maps all points to a particular line.

(ii) Find the line whose points all map to the point (3,1).

(iii) Without doing any calculations, what can be said about the line whose points all map to the point (6,2)?

(iv) Write down the line whose points all map to the Origin.

Solution

(i) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 6y \\ x + 2y \end{pmatrix} = \begin{pmatrix} 3(x + 2y) \\ x + 2y \end{pmatrix}$

So all points map to the line $y = \frac{1}{3}x$

(ii) $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow x + 2y = 1$

ie the required line is $y = -\frac{1}{2}x + \frac{1}{2}$

(iii) The line will have gradient $-\frac{1}{2}$ (lines mapping to different points cannot intersect (and therefore must be parallel) - otherwise the intersection point would map to two points).

[Note: The line doesn't contain the point (6,2).]

(iv) $y = -\frac{1}{2}x$ (It must contain the Origin, as this always maps to itself.)