Matrices - Exercises: General (Solutions)

(6 pages; 30/10/18)

(1) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Solution

Suppose that $\begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $af + bh = 0$ & $ce + dg = 0$

So $h = -\frac{af}{b}$ & $g = -\frac{ce}{d}$ (*)

Also $ae + bg = 1$ & $cf + dh = 1$,

so that $ae - \frac{bce}{d} = 1 \Rightarrow e(ad - bc) = d$

and $cf - \frac{daf}{b} = 1 \Rightarrow f(bc - ad) = b$

Let $\Delta = ad - bc$

Then $e = \frac{d}{\Delta}$ & $f = -\frac{b}{\Delta}$

And, from (*), $g = -\frac{c}{\Delta}$ & $h = \frac{a}{\Delta}$

Thus $\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

Solution

Define N^L to be the left inverse of N, so that $N^L N = I$

$NM = I$

$\Rightarrow N^L(NM) = N^L I = N^L$

$\Rightarrow (N^L N)M = N^L$

$\Rightarrow IM = N^L$

$\Rightarrow M = N^L$

$\Rightarrow MN = N^L N = I$

ie N is the right inverse of M

(3) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

Solution

Let $X = AB$

Then $XX^{-1} = I$, so that $ABX^{-1} = I$

Hence $A^{-1}ABX^{-1} = A^{-1}I$,

so that $BX^{-1} = A^{-1}$

Then $B^{-1}BX^{-1} = B^{-1}A^{-1}$,

so that $X^{-1} = B^{-1}A^{-1}$

(4) Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

Solution

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$

$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

and $x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$

$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

Hence $\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

and so $\gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

giving $\gamma^2 \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and hence $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

[This is the Lorentz factor.]

(5) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

Solution

Let $B = (A^T)^{-1}$, so that $BA^T = I$ (1)

Result to prove: $B = (A^{-1})^T$

[Noting that this is equivalent to $B^T = A^{-1}$, it seems promising to involve B^T]

From (1), $(BA^T)^T = I^T = I$, so that $AB^T = I$,

and hence $B^T = A^{-1}$ and $B = (A^{-1})^T$, as required.

(6)(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

Solution

(i) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{vmatrix} = (3k - 6) + (6 - 2) + (6 - k)$ [expanding by the 1st row]

$= 2k + 4$

The equations don't have a unique solution when $2k + 4 = 0$; ie $k = -2$

In that case, the equations are:

$x - y + z = 1$

$2x - 2y + 2z = 3$

$x + 3y + 3z = 5$

As the direction vectors of the first two planes are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, which are equivalent, and the constant terms on the RHS are not in the same ratio as the LHS terms, these planes are parallel, and the 3rd plane cuts both of the other planes (not being parallel to either of them).

(ii) To solve $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$: det $= 2k + 4 = 8$

and so $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 0 & -4 & 4 \\ 6 & 2 & -4 \\ -4 & 0 & 4 \end{pmatrix}^{T} =$
$\frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}$

[eg $6 = -((-1) \times 3 - 3 \times 1)$; $2 = 1 \times 3 - 1 \times 1$;

$-4 = -(1 \times 3 - 1 \times (-1))$]

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -2 \\ 2 \\ 12 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$

ie $x = \frac{-1}{4}$, $y = \frac{1}{4}$, $z = \frac{3}{2}$

(7) Factorise the determinant $\begin{vmatrix} x^2-x & y^2-y & z^2-z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Solution

$C2 \rightarrow C2 - C1$ & $C3 \rightarrow C3 - C1 \Rightarrow$

$$\begin{vmatrix} x^2-x & y^2-y-x^2+x & z^2-z-x^2+x \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y^2-x^2)-(y-x) & (z^2-x^2)-(z-x) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y-x)(y+x-1) & (z-x)(z+x-1) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x^2-x & y+x-1 & z+x-1 \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y+x-1-(z+x-1)\}$$

$$= (y-x)(z-x)(y-z)$$

Alternatively:

$R1 \rightarrow R1 + R2 \Rightarrow \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$C2 \rightarrow C2 - C1$ & $C3 \rightarrow C3 - C1 \Rightarrow \begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$

$$= (y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)$$