Matrices - Exercises: General (Solutions)

(10 pages; 13/8/19)

(1) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Solution

Suppose that $\begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $af + bh = 0$ & $ce + dg = 0$

So $h = -\frac{af}{b}$ & $g = -\frac{ce}{d}$ (*)

Also $ae + bg = 1$ & $cf + dh = 1$,

so that $ae - \frac{bce}{d} = 1 \Rightarrow e(ad - bc) = d$

and $cf - \frac{daf}{b} = 1 \Rightarrow f(bc - ad) = b$

Let $\Delta = ad - bc$

Then $e = \frac{d}{\Delta}$ & $f = -\frac{b}{\Delta}$

And, from (*), $g = -\frac{c}{\Delta}$ & $h = \frac{a}{\Delta}$

Thus $\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

Solution

Define N^L to be the left inverse of N, so that $N^L N = I$

$NM = I$

$\Rightarrow N^L(NM) = N^L I = N^L$

$\Rightarrow (N^L N)M = N^L$

$\Rightarrow IM = N^L$

$\Rightarrow M = N^L$

$\Rightarrow MN = N^L N = I$

ie N is the right inverse of M

(3) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

Solution

Let $X = AB$

Then $XX^{-1} = I$, so that $ABX^{-1} = I$

Hence $A^{-1}ABX^{-1} = A^{-1}I$,

so that $BX^{-1} = A^{-1}$

Then $B^{-1}BX^{-1} = B^{-1}A^{-1}$,

so that $X^{-1} = B^{-1}A^{-1}$

(4) Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

Solution

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$

$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

and $x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$

$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

Hence $\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

and so $\gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

giving $\gamma^2 \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and hence $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

[This is the Lorentz factor.]

(5) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

Solution

Let $B = (A^T)^{-1}$, so that $BA^T = I$ (1)

Result to prove: $B = (A^{-1})^T$

[Noting that this is equivalent to $B^T = A^{-1}$, it seems promising to involve B^T]

From (1), $(BA^T)^T = I^T = I$, so that $AB^T = I$,

and hence $B^T = A^{-1}$ and $B = (A^{-1})^T$, as required.

(6)(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

Solution

(i) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{vmatrix} = (3k - 6) + (6 - 2) + (6 - k)$ [expanding by the 1st row]

$= 2k + 4$

The equations don't have a unique solution when $2k + 4 = 0$; ie $k = -2$

In that case, the equations are:

$x - y + z = 1$

$2x - 2y + 2z = 3$

$x + 3y + 3z = 5$

As the direction vectors of the first two planes are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, which are equivalent, and the constant terms on the RHS are not in the same ratio as the LHS terms, these planes are parallel, and the 3rd plane cuts both of the other planes (not being parallel to either of them).

(ii) To solve $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$: det $= 2k + 4 = 8$

and so $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 0 & -4 & 4 \\ 6 & 2 & -4 \\ -4 & 0 & 4 \end{pmatrix}^{T} =$
$\frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}$

[eg $6 = -\left((-1) \times 3 - 3 \times 1\right); 2 = 1 \times 3 - 1 \times 1;$
$-4 = -\left(1 \times 3 - 1 \times (-1)\right)$]

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -2 \\ 2 \\ 12 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$

ie $x = \frac{-1}{4}$, $y = \frac{1}{4}$, $z = \frac{3}{2}$

(7) Factorise the determinant $\begin{vmatrix} x^2-x & y^2-y & z^2-z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Solution

$C2 \to C2 - C1$ & $C3 \to C3 - C1 \Rightarrow$

$$\begin{vmatrix} x^2-x & y^2-y-x^2+x & z^2-z-x^2+x \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y^2-x^2)-(y-x) & (z^2-x^2)-(z-x) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y-x)(y+x-1) & (z-x)(z+x-1) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x^2-x & y+x-1 & z+x-1 \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y+x-1-(z+x-1)\}$$

$$= (y-x)(z-x)(y-z)$$

Alternatively:

$R1 \to R1 + R2 \Rightarrow \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$C2 \to C2 - C1$ & $C3 \to C3 - C1 \Rightarrow \begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$

$$= (y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)$$

(8) Find the value of k for which the following equations are consistent.

$3x - 3y - z = k$

$2x - y - z = 5$

$x + 4y - 2z = 7$

Solution

$3x - 3y - z = k$ (1)

$2x - y - z = 5$ (2)

$x + 4y - 2z = 7$ (3)

Method 1

Using (2) to eliminate z in (1) & (3):

$3x - 3y - (2x - y - 5) = k$; ie $x - 2y = k - 5$ (1')

$x + 4y - 2(2x - y - 5) = 7$; ie $-3x + 6y = -3$

and $x - 2y = 1$ (3')

Hence, $k - 5 = 1$ for consistency, so that $k = 6$

Method 2

$$\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 3(6) - 2(10) + 1(2) = 0$$

By Cramer's rule, $x = \frac{\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix}}$, and this will only have a value if

$$\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix} = 0$$

ie when $k(6) - 5(10) + 7(2) = 0$,

so that $6k = 36; k = 6$

(9) Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$x - y + 3z = 4$
$4x + 5y - 2z = 8$
$x + 17y - 25z = -12$

Solution

First of all, none of the lines are parallel to each other.

Then $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \\ 1 & 17 & -25 \end{vmatrix} = 1(-91) - (-1)(-98) + 3(63) = 0$

[as expected for this sort of question]

So the planes will either be configured as a sheaf (if they have a line of intersection) or as a triangular prism (if not).

[In some cases it may be possible to spot that one equation is a combination of the other two, showing that the equations are consistent, and that they meet in a line.]

$x - y + 3z = 4 \quad (1)$
$4x + 5y - 2z = 8\ (2)$
$x + 17y - 25z = -12(3)$

Substituting for x (say), from (1) into (2) gives:

$4(4 + y - 3z) + 5y - 2z = 8$, so that $9y - 14z = -8$

Substituting into (3) gives:

$(4 + y - 3z) + 17y - 25z = -12$, so that $18y - 28z = -16$,

which is the same equation, and hence the planes meet as a sheaf.

To find the line of intersection, let $x = \lambda$ (say).

Then, from (1), $-y + 3z = 4 - \lambda$ (3)

and from (2), $5y - 2z = 8 - 4\lambda$ (4)

Then $5(3) + (4) \Rightarrow 13z = 28 - 9\lambda$

and $2(3) + 3(4) \Rightarrow 13y = 32 - 14\lambda$,

so that the equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13\lambda \\ 32 - 14\lambda \\ 28 - 9\lambda \end{pmatrix}$

or $\boldsymbol{\frac{x}{13} = \frac{y - \frac{32}{13}}{-14} = \frac{z - \frac{28}{13}}{-9}}$

[As a check, points on the line where $\lambda = 0$ and 1 could be substituted into the equations of the planes.

Also, it can be shown that the determinant formed by replacing (any) one of the columns of the matrix by the right-hand values will be zero when the equations are consistent. (Consider the 2×2 case to see why this is likely to be true.)

Thus $\begin{vmatrix} 1 & -1 & 4 \\ 4 & 5 & 8 \\ 1 & 17 & -12 \end{vmatrix} = 1(-196) - (-1)(-56) + 4(63) = 0$, for example.]

(10) Write the determinant $\begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$ as a product of linear factors.

Solution

Replacing row 1 with row 1 - row 2,

$$D = \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix} = \begin{vmatrix} 0 & x^2 - y^2 & x^4 - y^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2)\begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

Similarly, replacing row 2 with row 2 - row 3,

$$D = (x^2 - y^2)(y^2 - z^2)\begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 0 & 1 & y^2 + z^2 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2)(y^2 - z^2)(y^2 + z^2 - [x^2 + y^2])$$

$$= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

$$= (x - y)(x + y)(y - z)(y + z)(z - x)(z + x)$$