Matrices - Exercises: General (Solutions)

(14 pages; 10/1/20)

(1*) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Solution

Suppose that $\begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $af + bh = 0$ & $ce + dg = 0$

So $h = -\frac{af}{b}$ & $g = -\frac{ce}{d}$ (*)

Also $ae + bg = 1$ & $cf + dh = 1$,

so that $ae - \frac{bce}{d} = 1 \Rightarrow e(ad - bc) = d$

and $cf - \frac{daf}{b} = 1 \Rightarrow f(bc - ad) = b$

Let $\Delta = ad - bc$

Then $e = \frac{d}{\Delta}$ & $f = -\frac{b}{\Delta}$

And, from (*), $g = -\frac{c}{\Delta}$ & $h = \frac{a}{\Delta}$

Thus $\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2***) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

Solution

Define N^L to be the left inverse of N, so that $N^L N = I$

$NM = I$

$\Rightarrow N^L(NM) = N^L I = N^L$

$\Rightarrow (N^L N)M = N^L$

$\Rightarrow IM = N^L$

$\Rightarrow M = N^L$

$\Rightarrow MN = N^L N = I$

ie N is the right inverse of M

(3**) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

Solution

Let $X = AB$

Then $XX^{-1} = I$, so that $ABX^{-1} = I$

Hence $A^{-1}ABX^{-1} = A^{-1}I$,

so that $BX^{-1} = A^{-1}$

Then $B^{-1}BX^{-1} = B^{-1}A^{-1}$,

so that $X^{-1} = B^{-1}A^{-1}$

(4***) Suppose that the following pair of equations enables (x', t') to be determined from (x, t):

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of v. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for γ in terms of v & c.

Solution

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$

$\Rightarrow \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

and $x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$

$\Rightarrow \begin{pmatrix} x \\ t \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

Hence $\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} \begin{pmatrix} x' \\ t' \end{pmatrix}$

and so $\gamma \begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

giving $\gamma^2 \begin{pmatrix} 1 - \frac{v^2}{c^2} & 0 \\ 0 & 1 - \frac{v^2}{c^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and hence $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

[This is the Lorentz factor.]

(5***) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

Solution

Let $B = (A^T)^{-1}$, so that $BA^T = I$ (1)

Result to prove: $B = (A^{-1})^T$

[Noting that this is equivalent to $B^T = A^{-1}$, it seems promising to involve B^T]

From (1), $(BA^T)^T = I^T = I$, so that $AB^T = I$,

and hence $B^T = A^{-1}$ and $B = (A^{-1})^T$, as required.

(6***)(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of k do the planes not meet at a single point? For this value of k how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

Solution

(i) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & k & 2 \\ 1 & 3 & 3 \end{vmatrix} = (3k - 6) + (6 - 2) + (6 - k)$ [expanding by the 1st row]

$= 2k + 4$

The equations don't have a unique solution when $2k + 4 = 0$; ie $k = -2$

In that case, the equations are:

$x - y + z = 1$

$2x - 2y + 2z = 3$

$x + 3y + 3z = 5$

As the direction vectors of the first two planes are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$, which are equivalent, and the constant terms on the RHS are not in the same ratio as the LHS terms, these planes are parallel, and the 3rd plane cuts both of the other planes (not being parallel to either of them).

(ii) To solve $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$: det $= 2k + 4 = 8$

and so $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 0 & -4 & 4 \\ 6 & 2 & -4 \\ -4 & 0 & 4 \end{pmatrix}^{T} =$
$\frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}$

[eg $6 = -\big((-1) \times 3 - 3 \times 1\big)$; $2 = 1 \times 3 - 1 \times 1$;
$-4 = -\big(1 \times 3 - 1 \times (-1)\big)$]

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 6 & -4 \\ -4 & 2 & 0 \\ 4 & -4 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -2 \\ 2 \\ 12 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$

ie $x = \frac{-1}{4}$, $y = \frac{1}{4}$, $z = \frac{3}{2}$

(7***) Factorise the determinant $\begin{vmatrix} x^2-x & y^2-y & z^2-z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Solution

$C2 \to C2 - C1 \;\&\; C3 \to C3 - C1 \Rightarrow$

$$\begin{vmatrix} x^2-x & y^2-y-x^2+x & z^2-z-x^2+x \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y^2-x^2)-(y-x) & (z^2-x^2)-(z-x) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x^2-x & (y-x)(y+x-1) & (z-x)(z+x-1) \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x^2-x & y+x-1 & z+x-1 \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)\{y+x-1-(z+x-1)\}$$

$$= (y-x)(z-x)(y-z)$$

Alternatively:

$R1 \to R1 + R2 \Rightarrow \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

$C2 \to C2 - C1 \;\&\; C3 \to C3 - C1 \Rightarrow \begin{vmatrix} x^2 & y^2-x^2 & z^2-x^2 \\ x & y-x & z-x \\ 1 & 0 & 0 \end{vmatrix}$

$$= (y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ x & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (y-x)(z-x)(y-z)$$

(8***) Find the value of k for which the following equations are consistent.

$3x - 3y - z = k$

$2x - y - z = 5$

$x + 4y - 2z = 7$

Solution

$3x - 3y - z = k$ (1)

$2x - y - z = 5$ (2)

$x + 4y - 2z = 7$ (3)

Method 1

Using (2) to eliminate z in (1) & (3):

$3x - 3y - (2x - y - 5) = k$; ie $x - 2y = k - 5$ (1')

$x + 4y - 2(2x - y - 5) = 7$; ie $-3x + 6y = -3$

and $x - 2y = 1$ (3')

Hence, $k - 5 = 1$ for consistency, so that $k = 6$

Method 2

$$\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix} = 3(6) - 2(10) + 1(2) = 0$$

By Cramer's rule, $x = \dfrac{\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -3 & -1 \\ 2 & -1 & -1 \\ 1 & 4 & -2 \end{vmatrix}}$, and this will only have a value if

$$\begin{vmatrix} k & -3 & -1 \\ 5 & -1 & -1 \\ 7 & 4 & -2 \end{vmatrix} = 0$$

ie when $k(6) - 5(10) + 7(2) = 0$,

so that $6k = 36$; $k = 6$

(9***) Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$x - y + 3z = 4$
$4x + 5y - 2z = 8$
$x + 17y - 25z = -12$

Solution

First of all, none of the lines are parallel to each other.

Then $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \\ 1 & 17 & -25 \end{vmatrix} = 1(-91) - (-1)(-98) + 3(63) = 0$

[as expected for this sort of question]

So the planes will either be configured as a sheaf (if they have a line of intersection) or as a triangular prism (if not).

[In some cases it may be possible to spot that one equation is a combination of the other two, showing that the equations are consistent, and that they meet in a line.]

$x - y + 3z = 4 \quad (1)$
$4x + 5y - 2z = 8 \ (2)$
$x + 17y - 25z = -12 (3)$

Substituting for x (say), from (1) into (2) gives:

$4(4 + y - 3z) + 5y - 2z = 8$, so that $9y - 14z = -8$

Substituting into (3) gives:

$(4 + y - 3z) + 17y - 25z = -12$, so that $18y - 28z = -16$,

which is the same equation, and hence the planes meet as a sheaf.

To find the line of intersection, let $x = \lambda$ (say).

Then, from (1), $-y + 3z = 4 - \lambda$ (3)

and from (2), $5y - 2z = 8 - 4\lambda$ (4)

Then $5(3) + (4) \Rightarrow 13z = 28 - 9\lambda$

and $2(3) + 3(4) \Rightarrow 13y = 32 - 14\lambda$,

so that the equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 13\lambda \\ 32 - 14\lambda \\ 28 - 9\lambda \end{pmatrix}$

or $\frac{x}{13} = \frac{y - \frac{32}{13}}{-14} = \frac{z - \frac{28}{13}}{-9}$

[As a check, points on the line where $\lambda = 0$ and 1 could be substituted into the equations of the planes.

Also, it can be shown that the determinant formed by replacing (any) one of the columns of the matrix by the right-hand values will be zero when the equations are consistent. (Consider the 2×2 case to see why this is likely to be true.)

Thus $\begin{vmatrix} 1 & -1 & 4 \\ 4 & 5 & 8 \\ 1 & 17 & -12 \end{vmatrix} = 1(-196) - (-1)(-56) + 4(63) = 0$, for example.]

(10***) Write the determinant $\begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$ as a product of linear factors.

Solution

Replacing row 1 with row 1 - row 2,

$$D = \begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix} = \begin{vmatrix} 0 & x^2 - y^2 & x^4 - y^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2) \begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$$

Similarly, replacing row 2 with row 2 - row 3,

$$D = (x^2 - y^2)(y^2 - z^2) \begin{vmatrix} 0 & 1 & x^2 + y^2 \\ 0 & 1 & y^2 + z^2 \\ 1 & z^2 & z^4 \end{vmatrix}$$

$$= (x^2 - y^2)(y^2 - z^2)(y^2 + z^2 - [x^2 + y^2])$$

$$= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$$

$$= (x - y)(x + y)(y - z)(y + z)(z - x)(z + x)$$

(11****) Find the condition(s) for two 2×2 matrices to commute.

Solution

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\Rightarrow ae + cf = ae + bg \Rightarrow \frac{b}{c} = \frac{f}{g}$ (1)

Also $bg + dh = cf + dh \Rightarrow$ same condition

Then $be + df = af + bh$ (2) and $ag + ch = ce + dg$ (3)

(2) $\Rightarrow b(e-h) = f(a-d)$ and (3) $\Rightarrow c(h-e) = g(d-a)$

From (1), $\frac{b}{f} = \frac{c}{g}$ and so both of the above produce the same condition:

$\frac{b}{f} = \frac{a-d}{e-h} \Rightarrow \frac{a-d}{b} = \frac{e-h}{f}$ (4)

Thus, two 2×2 matrices commute if the quantities $\frac{b}{c}$ and $\frac{a-d}{b}$ in one matrix match the corresponding quantities in the other.

As an example, we could choose the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & g \\ 6 & h \end{pmatrix}$.

Then $g = 6 \times \frac{3}{2} = 9$ and $\frac{h-5}{6} = \frac{4-1}{2} \Rightarrow h = 14$

Check: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

and $\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

To test the conditions on a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and its inverse,

$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(i) $\frac{-b/(ad-bc)}{-c/(ad-bc)} = \frac{b}{c}$

(ii) $\frac{(d-a)/(ad-bc)}{-b/(ad-bc)} = \frac{a-d}{b}$

(12****) Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

Solution

From $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, if $R1 \to R1 - a(R2)$, we get $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$

Then, $C3 \to C3 - c(C2)$ gives $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

and finally $R1 \to R1 - (b-ac)(R3)$ gives $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

[Note that no further factors have had to be taken outside the determinant - as expected, since $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = 1$]

(13****) Show that a matrix is orthogonal if and only if

(i) its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero), and
(ii) each column has unit magnitude

Solution

A matrix P is orthogonal when $P^{-1} = P^T$; ie when $PP^T = I$

Suppose that $P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, so that $PP^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Then the diagonal entries will be 1 when the 3 columns of P have unit magnitude, and the non-diagonal entries will be 0 when the columns are mutually orthogonal.

(14****) Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]

Solution

As the position vector $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is in the plane containing the Origin and the position vectors $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, it follows that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is perpendicular to the normal to that plane; ie perpendicular to

$\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & -1 & 0 \\ \underline{j} & 0 & 2 \\ \underline{k} & 3 & 4 \end{vmatrix}$; so that

$\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} . \begin{vmatrix} \underline{i} & -1 & 0 \\ \underline{j} & 0 & 2 \\ \underline{k} & 3 & 4 \end{vmatrix} = 0$, and thus $\begin{vmatrix} 2 & -1 & 0 \\ 3 & 0 & 2 \\ c & 3 & 4 \end{vmatrix} = 0$

Alternative Approach 1

The 3 vectors form a parallelepiped of zero volume, so that the scalar triple product of the vectors is zero.]

Alternative Approach 2

The required relation can be written as

$$\begin{pmatrix}2\\3\\c\end{pmatrix}-a\begin{pmatrix}-1\\0\\3\end{pmatrix}-b\begin{pmatrix}0\\2\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},$$

which implies a solution of $x\begin{pmatrix}2\\3\\c\end{pmatrix}+y\begin{pmatrix}-1\\0\\3\end{pmatrix}+z\begin{pmatrix}0\\2\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

other than $x=y=z=0$,

and for there to be more than one solution to this matrix equation, we require the determinant to be zero. ■

Then $\begin{vmatrix}2 & -1 & 0\\3 & 0 & 2\\c & 3 & 4\end{vmatrix}=0 \Rightarrow 2(-6)-(-1)(12-2c)=0$

$\Rightarrow c=0$

$$\begin{pmatrix}2\\3\\0\end{pmatrix}=a\begin{pmatrix}-1\\0\\3\end{pmatrix}+b\begin{pmatrix}0\\2\\4\end{pmatrix}$$

$\Rightarrow 2=-a$

$3=2b$

& $0=3a+4b$

so that $a=-2$ & $b=\frac{3}{2}$