Matrices - Exercises: General - Advanced (Solutions)

(5 pages; 30/10/18)

(1) Find the condition(s) for two 2×2 matrices to commute.

Solution

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\Rightarrow ae + cf = ae + bg \Rightarrow \frac{b}{c} = \frac{f}{g}$ (1)

Also $bg + dh = cf + dh \Rightarrow$ same condition

Then $be + df = af + bh$ (2) and $ag + ch = ce + dg$ (3)

(2) $\Rightarrow b(e - h) = f(a - d)$ and (3) $\Rightarrow c(h - e) = g(d - a)$

From (1), $\frac{b}{f} = \frac{c}{g}$ and so both of the above produce the same condition:

$\frac{b}{f} = \frac{a-d}{e-h} \Rightarrow \frac{a-d}{b} = \frac{e-h}{f}$ (4)

Thus, two 2×2 matrices commute if the quantities $\frac{b}{c}$ and $\frac{a-d}{b}$ in one matrix match the corresponding quantities in the other.

As an example, we could choose the matrices $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $\begin{pmatrix} 5 & g \\ 6 & h \end{pmatrix}$.

Then $g = 6 \times \frac{3}{2} = 9$ and $\frac{h-5}{6} = \frac{4-1}{2} \Rightarrow h = 14$

Check: $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

and $\begin{pmatrix} 5 & 9 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 51 \\ 34 & 74 \end{pmatrix}$

To test the conditions on a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and its inverse,

$\frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(i) $\frac{-b/(ad-bc)}{-c/(ad-bc)} = \frac{b}{c}$

(ii) $\frac{(d-a)/(ad-bc)}{-b/(ad-bc)} = \frac{a-d}{b}$

(2) Given that a 3×3 determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

Solution

From $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, if $R1 \to R1 - a(R2)$, we get $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$

Then, $C3 \to C3 - c(C2)$ gives $\begin{vmatrix} 1 & 0 & b-ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

and finally $R1 \to R1 - (b-ac)(R3)$ gives $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

[Note that no further factors have had to be taken outside the determinant - as expected, since $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix} = 1$]

(3) Show that a matrix is orthogonal if and only if

(i) its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero), and
(ii) each column has unit magnitude

Solution

A matrix P is orthogonal when $P^{-1} = P^T$; ie when $PP^T = I$

Suppose that $P = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$,

so that $PP^T = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

Then the diagonal entries will be 1 when the 3 columns of P have unit magnitude, and the non-diagonal entries will be 0 when the columns are mutually orthogonal.

(4) Find c, a & b such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]

Solution

As the position vector $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is in the plane containing the Origin and the position vectors $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$, it follows that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}$ is perpendicular to the normal to that plane; ie perpendicular to

$\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & -1 & 0 \\ \underline{j} & 0 & 2 \\ \underline{k} & 3 & 4 \end{vmatrix}$; so that

$$\begin{pmatrix}2\\3\\c\end{pmatrix}.\begin{vmatrix}\underline{i} & -1 & 0\\ \underline{j} & 0 & 2\\ \underline{k} & 3 & 4\end{vmatrix} = 0 \text{ , and thus } \begin{vmatrix}2 & -1 & 0\\ 3 & 0 & 2\\ c & 3 & 4\end{vmatrix} = 0$$

Alternative Approach 1

The 3 vectors form a parallelepiped of zero volume, so that the scalar triple product of the vectors is zero.]

Alternative Approach 2

The required relation can be written as

$$\begin{pmatrix}2\\3\\c\end{pmatrix} - a\begin{pmatrix}-1\\0\\3\end{pmatrix} - b\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

which implies a solution of $x\begin{pmatrix}2\\3\\c\end{pmatrix} + y\begin{pmatrix}-1\\0\\3\end{pmatrix} + z\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

other than $x = y = z = 0$,

and for there to be more than one solution to this matrix equation, we require the determinant to be zero. ■

Then $\begin{vmatrix}2 & -1 & 0\\ 3 & 0 & 2\\ c & 3 & 4\end{vmatrix} = 0 \Rightarrow 2(-6) - (-1)(12 - 2c) = 0$

$\Rightarrow c = 0$

$$\begin{pmatrix}2\\3\\0\end{pmatrix} = a\begin{pmatrix}-1\\0\\3\end{pmatrix} + b\begin{pmatrix}0\\2\\4\end{pmatrix}$$

$\Rightarrow 2 = -a$

$3 = 2b$

& $0 = 3a + 4b$

so that $a = -2$ & $b = \frac{3}{2}$