# Matrices - Exercises: General (3 pages; 10/1/20)

**Key to difficulty:**

* introductory exercise

** light A Level (FM) standard

*** harder A Level (FM) standard

**** harder than A Level (FM)

(1*) Prove that $\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

(2***) Show that if N is the left inverse of M, so that $NM = I$, then it is also the right inverse.

(3**) Prove that $(AB)^{-1} = B^{-1}A^{-1}$

(4***) Suppose that the following pair of equations enables $(x', t')$ to be determined from $(x, t)$:

$x' = \gamma(x - vt)$ & $t' = \gamma(t - \frac{xv}{c^2})$ (A)

and that it is also true that

$x = \gamma(x' + vt')$ & $t = \gamma(t' + \frac{x'v}{c^2})$ (B)

[These are the transformation equations in Special Relativity between two frames of reference that are moving with a relative speed of $v$. Starting with (A), (B) is obtained by reversing the roles of the two frames (so that the speed is reversed as well).]

Use matrix multiplication to find an expression for $\gamma$ in terms of $v$ & $c$.

(5***) Assuming that $(AB)^T = B^T A^T$, prove that $(A^T)^{-1} = (A^{-1})^T$

(6***)(i) Three planes are represented by the following equations:

$x - y + z = 1$

$2x + ky + 2z = 3$

$x + 3y + 3z = 5$

For what value of $k$ do the planes not meet at a single point? For this value of $k$ how are the planes configured?

(ii) If $k = 2$, find the point of intersection, using matrices.

(7***) Factorise the determinant $\begin{vmatrix} x^2 - x & y^2 - y & z^2 - z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

(8***) Find the value of $k$ for which the following equations are consistent.

$3x - 3y - z = k$

$2x - y - z = 5$

$x + 4y - 2z = 7$

(9***) Show that the following three planes meet in a line, giving the equation of that line in cartesian form.

$x - y + 3z = 4$
$4x + 5y - 2z = 8$
$x + 17y - 25z = -12$

(10***) Write the determinant $\begin{vmatrix} 1 & x^2 & x^4 \\ 1 & y^2 & y^4 \\ 1 & z^2 & z^4 \end{vmatrix}$ as a product of linear factors.

(11****) Find the condition(s) for two $2 \times 2$ matrices to commute.

(12****) Given that a $3 \times 3$ determinant can always be reduced to triangular form (in the same way as simultaneous equations), to produce a multiple of $\begin{vmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$, show that it can be further reduced to a multiple of the Identity matrix. [Obviously this is an academic exercise, as the determinant can be evaluated as soon as triangular form has been reached.]

(13****) Show that a matrix is orthogonal if and only if

(i) its columns are mutually orthogonal (ie perpendicular, so that their scalar product is zero), and
(ii) each column has unit magnitude

(14****) Find $c, a$ & $b$ such that $\begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} = a\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

[ie such that the 3 vectors are not linearly independent]