Matrices - Exercises: Eigenvectors (Solutions)

(12 pages; 30/10/18)

(1) Find $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3$, using eigenvectors.

Solution

To find the eigenvalues of $M = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$:

[We want a non-zero solution of $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$;

ie of $(M - \lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$; for there to be more than one solution (ie a non-zero solution, as well as the zero solution), $|M - \lambda I| = 0$;
ie $\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$]

The characteristic equation is $\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$,

so that $(2-\lambda)(3-\lambda) - 2 = 0$ and $\lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 4) = 0$

Thus the eigenvalues are $\lambda = 1$ and 4

The eigenvectors satisfy $\begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For $\lambda = 1$: $x + y = 0$; $2x + 2y = 0$

[as a check, these equations should be equivalent, and so producing more than one solution]

Thus an eigenvector for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $\lambda = 4$: $-2x + y = 0$; $2x - y = 0$

Thus an eigenvector for $\lambda = 4$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

[Note: any multiples of these eigenvectors are also solutions, and so there is an infinite number of solutions]

Let $S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Then $MS = SD$

[considering the columns of S separately, and noting that $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and so $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; thus $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and similarly for the 1st column]

and hence $M = SDS^{-1}$

so that $M^3 = (SDS^{-1})(SDS^{-1})(SDS^{-1}) = SD^3S^{-1}$

$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 1 & 64 \\ -1 & 128 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 66 & 63 \\ 126 & 129 \end{pmatrix} = \begin{pmatrix} 22 & 21 \\ 42 & 43 \end{pmatrix}$

[we would obviously expect to have only integers in the answer, being a power of M]

Check: $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 10 & 11 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 22 & 21 \\ 42 & 43 \end{pmatrix}$

(2) If matrices M & N (both square, of the same order) share an eigenvector, what can be said about the eigenvectors and eigenvalues of $MN\ and\ NM$?

Solution

Let $M\underline{x} = \lambda\underline{x}$ and $N\underline{x} = \mu\underline{x}$

Then $(MN)\underline{x} = M(N\underline{x}) = M(\mu\underline{x}) = \mu(M\underline{x}) = \mu\lambda\underline{x}$

Thus MN, and similarly NM, also have this same eigenvector, and the associated eigenvalue is the product of the corresponding eigenvalues for M & N.

(3) Given that the eigenvalues of $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ are 4, 4 and 1,

establish the geometrical significance of the eigenvectors.

Solution

First of all, the eigenvector associated with the eigenvalue of 1 will be a line of invariant points (through the Origin) [All eigenvectors are invariant lines through the Origin, and are lines of invariant points when the eigenvalue is 1.]

[When there are repeated eigenvalues, there will either be an invariant plane or an invariant line. When there aren't repeated eigenvalues, there can only be an invariant line. See "Matrices - notes".]

$$\begin{pmatrix} 3-4 & -1 & 1 \\ -1 & 3-4 & 1 \\ 1 & 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

giving $-x - y + z = 0$ (3 times)

This is the equation of a plane; ie the invariant plane of the transformation (all points map to another point in the plane.)

(4) (i) Show that the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ are 1 (repeated) and 2 (for example, by using row or column operations), and investigate the geometrical significance of the eigenvectors.

(ii) Construct another matrix with the same eigenvalues, and hence establish that the geometrical result in (i) does not hold in general.

Solution

The characteristic equation for the matrix is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0\,,$$

As the eigenvalues have been given, we could (if only for practice at manipulating determinants) look for row or column operations that produce the required factorisation.

For example, replacing column 2 with column 2 + column 3:

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1-\lambda & 2-\lambda \end{vmatrix} = 0\,,$$

so that $(1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

[Note that, if two 0s can be created in a row or column, then a factor can be taken out; whilst the presence of one 0 means that any common factor of the other two elements in the row or column can be taken out. For example, if two rows (or columns) of the matrix share two corresponding elements, then it will be possible to create two 0s, by subtracting one row from the other.]

As the 2nd and 3rd rows share two corresponding elements, we can replace row 3 with row 3 − row 2, to give:

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0\,,$$

and so $(1-\lambda)(1-\lambda)(2-\lambda)$, expanding by the 3rd row.

Thus the eigenvalues are 1 (repeated) and 2.

To find the eigenvectors:

$$\lambda = 1 \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that we have the plane $x - y + z = 0$

Thus any point in this plane will be mapped to itself (as the eigenvalue is 1); ie it is a plane of invariant points.

The plane can also be presented in the form $\underline{r} = \alpha\underline{a} + \beta\underline{b}$, where

$\underline{a}$ & $\underline{b}$ are any two independent vectors in the plane.

For example, $\underline{r} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Thus $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are two eigenvectors of the transformation (though they are not unique).

$$\lambda = 2 \Rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $-y + z = 0$, $x - 2y + z = 0$ & $x - y = 0$

Then let $x = \mu$ (for example), so that $y = \mu$ & $z = \mu$,

and we have the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, which just represents the usual invariant line through the Origin.

(ii) We want the characteristic equation to be

$(\lambda - 1)^2(\lambda - 2) = 0$

leading to $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$ (1)

We could then set up a suitably general characteristic equation such as $\begin{vmatrix} a-\lambda & 0 & c \\ 1 & -\lambda & 0 \\ 1 & b & -\lambda \end{vmatrix} = 0$ (2)

and equate coefficients with (1).

Thus (2) becomes $(a - \lambda)\lambda^2 + c(b + \lambda) = 0$ (3)

[A bit of experimentation is necessary in arriving at the determinant in (2), in order that (3) is sufficiently general.]

Then, equating the coefficients in (1) & (3) gives:

$a = 4, c = -5$ & $b = -\frac{2}{5}$

so that (2) becomes $\begin{vmatrix} 4-\lambda & 0 & -5 \\ 1 & -\lambda & 0 \\ 1 & -\frac{2}{5} & -\lambda \end{vmatrix} = 0$

[It is probably worth checking that this produces the required eigenvalues of 1 and 2.]

The eigenvectors corresponding to $\lambda = 1$ are then found from:

$$\begin{pmatrix} 3 & 0 & -5 \\ 1 & -1 & 0 \\ 1 & -\frac{2}{5} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that $3x - 5z = 0$, $x - y = 0$ & $x - \frac{2y}{5} - z = 0$

Then let $x = \mu$, so that $y = \mu$ & $z = \frac{3\mu}{5}$

Thus an eigenvector is $\begin{pmatrix}5\\5\\3\end{pmatrix}$, and we don't have the plane that was found in (i); only a line of invariant points (through the Origin).

[The (advanced) theory behind this is based on the following theorem: "The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity." The algebraic multiplicity is the number of times that the eigenvalue appears as a root of the characteristic equation. The geometric multiplicity is the dimension of the line or plane relating to the eigenvalue: so an invariant line means a geometric multiplicity of 1, whilst an invariant plane means a geometric multiplicity of 2. So, by this theorem, there have to be repeated eigenvalues in order for there to be an invariant plane, but if there are repeated eigenvalues it doesn't follow that there will be an invariant plane.]

(5) For a 3×3 matrix M, show that

(i) the product of the eigenvalues of M equals $\det M$

(ii) the sum of the eigenvalues equals the sum of the elements on the leading diagonal of M (from top left to bottom right; this sum is called the trace of M, or trM)

Solution

(i) The eigenvalues of M are the roots of $f(\lambda) = det(M - \lambda I) = 0$, considered as a cubic equation in λ.

$f(\lambda)$ can be written as $g(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_1)(\lambda - \lambda_1)$

(as the determinant will contain the term $-\lambda^3$)

Then we note that $f(0) = detM$ & $g(0) = \lambda_1\lambda_2\lambda_3$, so that the constant term of $f(\lambda) = g(\lambda)$ is $detM = \lambda_1\lambda_2\lambda_3$.

(ii) $\lambda_1 + \lambda_2 + \lambda_3 = -\frac{b}{a}$, where a & b are the coefficients of λ^3 & λ^2

in $det(M - \lambda I) = \begin{vmatrix} c-\lambda & f & i \\ d & g-\lambda & j \\ e & h & k-\lambda \end{vmatrix}$

The only terms involving λ^2 are contained in

$(c-\lambda)(g-\lambda)(k-\lambda)$, and $b = c + g + k$

Then, as $a = -1$, $\lambda_1 + \lambda_2 + \lambda_3 = c + g + k$; ie trM.

(6) (i) If $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 & λ_3 of a 3×3 matrix M, prove that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) Deduce that a 3×3 matrix with distinct eigenvalues can always be diagonalised.

Proof

(i) Suppose that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are in fact coplanar, so that

$\underline{s}_3 = a\underline{s}_1 + \mathrm{b}\underline{s}_2$, where a & b are not both zero (1)

(by definition, eigenvectors are non-zero)

Then $\mathrm{M}\underline{s}_3 = aM\underline{s}_1 + \mathrm{bM}\underline{s}_2$ and hence $\lambda_3\underline{s}_3 = a\lambda_1\underline{s}_1 + \mathrm{b}\lambda_2\underline{s}_2$

Also, from (1), $\lambda_3\underline{s}_3 = a\lambda_3\underline{s}_1 + \mathrm{b}\lambda_3\underline{s}_2$,

so that $a\lambda_1\underline{s}_1 + \mathrm{b}\lambda_2\underline{s}_2 = a\lambda_3\underline{s}_1 + \mathrm{b}\lambda_3\underline{s}_2$

and hence $a(\lambda_1 - \lambda_3)\underline{s}_1 = \mathrm{b}(\lambda_3 - \lambda_2)\underline{s}_2$

But $\lambda_1 - \lambda_3$ & $\lambda_3 - \lambda_2$ are non-zero, and $\underline{s}_1$ & $\underline{s}_3$ are not parallel (as otherwise they would have the same eigenvalues), so that it must be the case that $a = b$, which contradicts (1).

Thus $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) From (i), as $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are not coplanar, the volume of the parallelepiped with sides $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ is non-zero; ie

$\underline{s}_1 . (\underline{s}_2 \times \underline{s}_3) \neq 0$, so that $|\underline{s}_1, \underline{s}_2, \underline{s}_3| \neq 0$, which means that the matrix $(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ has an inverse, and hence M can be diagonalised.

[Note that if there are repeated eigenvalues, then at least two of the columns of $(\underline{s}_1, \underline{s}_2, \underline{s}_3)$ will be identical, making

$|\underline{s}_1, \underline{s}_2, \underline{s}_3| = 0$.]

(7) Matrices A & B are said to be 'similar' if $B = PAP^{-1}$ for some matrix P (A need not be diagonal).

Prove that similar matrices have the same characteristic equation, and hence the same eigenvalues.

Solution

Let the characteristic equation of A be $\sum a_r \lambda^r = 0$, so that $\sum a_r A^r = 0$. Then $P(\sum a_r A^r) = 0$, so that $\sum P a_r A^r = 0$.

Then $(\sum P a_r A^r)P^{-1} = 0$, and hence $\sum P a_r A^r P^{-1} = 0$, so that

$\sum a_r P A^r P^{-1} = 0$.

As $B^r = (PAP^{-1})(PAP^{-1}) \dots = PA^r P^{-1}$, it follows that $\sum a_r B^r = 0$, and thus B has the same characteristic equation as A.

(8) Symmetric matrices are always diagonalisable. Prove that this is the case for 2×2 symmetric matrices.

Solution

Consider $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, with characteristic equation

$\begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0$

$\Leftrightarrow (a-\lambda)(c-\lambda)-b^2=0$

$\Leftrightarrow \lambda^2-(a+c)\lambda+ac-b^2=0$

The discriminant is $(a+c)^2-4(ac-b^2)=(a-c)^2+4b^2$,

which is always positive, assuming that a, b & c are not all zero.

So there will be 2 distinct eigenvalues, and hence 2 linearly independent eigenvectors. Thus M is diagonalisable.

(9) Prove that if M is orthogonally diagonalisable, then M is symmetric.

Solution

If M is orthogonally diagonalisable, then $M=PDP^{-1}$, where $P^{-1}=P^T$.

Then $M^T=(PDP^{-1})^T=(P^{-1})^TD^TP^T=PDP^{-1}=M$,

so that M is symmetric.

(10) Find the square roots of the matrix $\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$ in (2).

Solution

From (2), $\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}^{-1}$

By comparison with the method of finding $\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}^3$, we would expect the answer to be $\pm\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}^{-1}$.

To prove this though,

let $M=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$; $P=\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$ & $D=\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$,

so that $M=PDP^{-1}$.

Then let $M = A^2$ and $D = E^2$

[we want to show that $A = \pm PEP^{-1}$]

So $A^2 = PE^2P^{-1}$ (1)

Then $A = PEP^{-1}$ is a solution of (1), since

$(PEP^{-1})^2 = PEP^{-1}PEP^{-1} = PE^2P^{-1}$,

and $A = -PEP^{-1}$ is clearly also a solution of (1).

Thus we have found the two squares roots of M [though we have admittedly made the assumption that matrices only have two squares roots.]

So $A = \pm\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1}$

$= \pm\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \pm\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$

$= \pm\frac{1}{3}\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}$

Check: $\frac{1}{9}\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}^2 = \frac{1}{9}\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 18 & 9 \\ 18 & 27 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$

(11) Show that 2×2 matrices representing rotations are not diagonalisable.

Solution

A matrix representing a rotation can be expressed in the form

$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$

The characteristic equation for this matrix is

$\begin{vmatrix} cos\theta - \lambda & -sin\theta \\ sin\theta & cos\theta - \lambda \end{vmatrix} = 0$

$\Leftrightarrow (cos\theta - \lambda)^2 + sin^2\theta = 0$

$\Leftrightarrow \lambda^2 - 2cos\theta.\lambda + 1 = 0$

The discriminant is $4cos^2\theta - 4$, which is negative for positive θ.

Thus there are no eigenvalues, and hence the matrix cannot be diagonalised.

(12) For the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with eigenvalues λ_1 & λ_2, prove that $\lambda_1 + \lambda_2 = a + d$, and also that $\lambda_1\lambda_2 = |M|$

[this can be extended to 3×3 matrices]

Solution

The characteristic equation is $\begin{vmatrix} a - \lambda & c \\ b & d - \lambda \end{vmatrix} = 0$, so that

$(a - \lambda)(d - \lambda) - bc = 0$ and $\lambda^2 - (a + d)\lambda + ad - bc = 0$

and the roots λ_1& λ_2 satisfy $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = ad - bc$, as required.