Matrices - Exercises: Eigenvectors (3 pages; 10/1/20)

Key to difficulty:

* introductory exercise

** light A Level (FM) standard

*** harder A Level (FM) standard

**** harder than A Level (FM)

(1***) Find $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3$, using eigenvectors.

(2***) If matrices M & N (both square, of the same order) share an eigenvector, what can be said about the eigenvectors and eigenvalues of $MN\ and\ NM$?

(3***) Given that the eigenvalues of $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ are 4, 4 and 1, establish the geometrical significance of the eigenvectors.

(4****) (i) Show that the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ are 1 (repeated) and 2 (for example, by using row or column operations), and investigate the geometrical significance of the eigenvectors.

(ii) Construct another matrix with the same eigenvalues, and hence establish that the geometrical result in (i) does not hold in general.

(5***) For a 3×3 matrix M, show that

(i) the product of the eigenvalues of M equals $\det M$

(ii) the sum of the eigenvalues equals the sum of the elements on the leading diagonal of M (from top left to bottom right; this sum is called the trace of M, or trM)

(6****) (i) If $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 & λ_3 of a 3×3 matrix M, prove that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) Deduce that a 3×3 matrix with distinct eigenvalues can always be diagonalised.

(7***) Matrices A & B are said to be 'similar' if $B = PAP^{-1}$ for some matrix P (A need not be diagonal).

Prove that similar matrices have the same characteristic equation, and hence the same eigenvalues.

(8***) Symmetric matrices are always diagonalisable. Prove that this is the case for 2×2 symmetric matrices.

(9****) Prove that if M is orthogonally diagonalisable, then M is symmetric.

(10****) Find the square roots of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ in (2).

(11***) Show that 2×2 matrices representing rotations are not diagonalisable.

(12***) For the matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with eigenvalues λ_1 & λ_2, prove that $\lambda_1 + \lambda_2 = a + d$, and also that $\lambda_1 \lambda_2 = |M|$

[this can be extended to 3×3 matrices]

(13***) The populations of sparrows (x) and sparrowhawks (y) in a particular area satisfy the following differential equations:

$\frac{dx}{dt} = 0.1x - 2y$ and $\frac{dy}{dt} = 0.1x + y$

(where time is measured in years),

and initially there are 50 sparrows and 4 sparrowhawks.

The equations can be written as $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (*)

(i) Express $\begin{pmatrix} 0.1 & -2 \\ 0.1 & 1 \end{pmatrix}$ in the form $\boldsymbol{PDP^{-1}}$, where $\boldsymbol{D}$ is a diagonal matrix.

(ii) Show that (*) can be rewritten as as $\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \boldsymbol{D} \begin{pmatrix} u \\ v \end{pmatrix}$

(iii) Show that $\boldsymbol{u} = \boldsymbol{A}e^{0.6t}$ and $\boldsymbol{v} = \boldsymbol{B}e^{0.5t}$, where $\boldsymbol{A}$ **and** $\boldsymbol{B}$ are arbitrary constants, and hence solve the original differential equations.

(iv) What happens to the two populations?