Matrices – Q9: Eigenvectors [Problem/H](2/6/21)

(i) If $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are eigenvectors corresponding to distinct eigenvalues λ_1, λ_2 & λ_3 of a 3×3 matrix M, prove that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) Deduce that a 3×3 matrix with distinct eigenvalues can always be diagonalised.

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Solution

(i) Suppose that $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are in fact coplanar, so that

$\underline{s}_3 = a\underline{s}_1 + \mathrm{b}\underline{s}_2$, where a & b are not both zero (1)

(by definition, eigenvectors are non-zero)

Then $\mathrm{M}\underline{s}_3 = aM\underline{s}_1 + \mathrm{bM}\underline{s}_2$ and hence $\lambda_3\underline{s}_3 = a\lambda_1\underline{s}_1 + \mathrm{b}\lambda_2\underline{s}_2$

Also, from (1), $\lambda_3\underline{s}_3 = a\lambda_3\underline{s}_1 + \mathrm{b}\lambda_3\underline{s}_2$,

so that $a\lambda_1\underline{s}_1 + \mathrm{b}\lambda_2\underline{s}_2 = a\lambda_3\underline{s}_1 + \mathrm{b}\lambda_3\underline{s}_2$

and hence $a(\lambda_1 - \lambda_3)\underline{s}_1 = \mathrm{b}(\lambda_3 - \lambda_2)\underline{s}_2$

But $\lambda_1 - \lambda_3$ & $\lambda_3 - \lambda_2$ are non-zero, and $\underline{s}_1$ & $\underline{s}_3$ are not parallel (as otherwise they would have the same eigenvalues), so that it must be the case that $a = b$, which contradicts (1).

Thus $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ cannot be coplanar.

(ii) From (i), as $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ are not coplanar, the volume of the parallelepiped with sides $\underline{s}_1, \underline{s}_2$ & $\underline{s}_3$ is non-zero; ie

$\underline{s}_1 . (\underline{s}_2 \times \underline{s}_3) \neq 0$, so that $|\underline{s}_1, \underline{s}_2 , \underline{s}_3| \neq 0$, which means that the matrix $(\underline{s}_1, \underline{s}_2 , \underline{s}_3)$ has an inverse, and hence M can be diagonalised.

[Note that if there are repeated eigenvalues, then at least two of the columns of $(\underline{s}_1, \underline{s}_2\ , \underline{s}_3)$ will be identical, making

$\left|\underline{s}_1, \underline{s}_2\ , \underline{s}_3\right| = 0$.]