Matrices – Q8: Eigenvectors [Problem/H](2/6/21)

For a 3×3 matrix M, show that

(i) the product of the eigenvalues of M equals $\det M$

(ii) the sum of the eigenvalues equals the sum of the elements on the leading diagonal of M (from top left to bottom right; this sum is called the trace of M, or trM)

For a 3×3 matrix M, show that

(i) the product of the eigenvalues of M equals $\det M$

(ii) the sum of the eigenvalues equals the sum of the elements on the leading diagonal of M (from top left to bottom right; this sum is called the trace of M, or trM)

Solution

(i) The eigenvalues of M are the roots of $f(\lambda) = det(M - \lambda I) = 0$, considered as a cubic equation in λ.

$f(\lambda)$ can be written as $g(\lambda) = -(\lambda - \lambda_1)(\lambda - \lambda_1)(\lambda - \lambda_1)$

(as the determinant will contain the term $-\lambda^3$)

Then we note that $f(0) = detM$ & $g(0) = \lambda_1\lambda_2\lambda_3$, so that the constant term of $f(\lambda) = g(\lambda)$ is $detM = \lambda_1\lambda_2\lambda_3$.

(ii) $\lambda_1 + \lambda_2 + \lambda_3 = -\frac{b}{a}$, where a & b are the coefficients of λ^3 & λ^2 in $det(M - \lambda I) = \begin{vmatrix} c-\lambda & f & i \\ d & g-\lambda & j \\ e & h & k-\lambda \end{vmatrix}$

The only terms involving λ^2 are contained in

$(c-\lambda)(g-\lambda)(k-\lambda)$, and $b = c + g + k$

Then, as $a = -1$, $\lambda_1 + \lambda_2 + \lambda_3 = c + g + k$; ie trM.