Matrices – Q7: Eigenvectors [Problem/H](2/6/21)

(i) Show that the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ are 1 (repeated) and 2 (for example, by using row or column operations), and investigate the geometrical significance of the eigenvectors.

(ii) Construct another matrix with the same eigenvalues, and hence establish that the geometrical result in (i) does not hold in general.

(i) Show that the eigenvalues of the matrix $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ are 1 (repeated) and 2 (for example, by using row or column operations), and investigate the geometrical significance of the eigenvectors.

(ii) Construct another matrix with the same eigenvalues, and hence establish that the geometrical result in (i) does not hold in general.

Solution

The characteristic equation for the matrix is

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0\,,$$

As the eigenvalues have been given, we could (if only for practice at manipulating determinants) look for row or column operations that produce the required factorisation.

For example, replacing column 2 with column 2 + column 3:

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1-\lambda & 2-\lambda \end{vmatrix} = 0\,,$$

so that $(1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$

[Note that, if two 0s can be created in a row or column, then a factor can be taken out; whilst the presence of one 0 means that any common factor of the other two elements in the row or column can be taken out. For example, if two rows (or columns) of the matrix share two corresponding elements, then it will be possible to create two 0s, by subtracting one row from the other.]

As the 2nd and 3rd rows share two corresponding elements, we can replace row 3 with row 3 − row 2, to give:

$$(1-\lambda)\begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0\,,$$

and so $(1-\lambda)(1-\lambda)(2-\lambda)$, expanding by the 3rd row.

Thus the eigenvalues are 1 (repeated) and 2.

To find the eigenvectors:

$$\lambda = 1 \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that we have the plane $x - y + z = 0$

Thus any point in this plane will be mapped to itself (as the eigenvalue is 1); ie it is a plane of invariant points.

The plane can also be presented in the form $\underline{r} = \alpha\underline{a} + \beta\underline{b}$, where

$\underline{a}$ & $\underline{b}$ are any two independent vectors in the plane.

For example, $\underline{r} = \alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Thus $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are two eigenvectors of the transformation (though they are not unique).

$$\lambda = 2 \Rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $-y + z = 0$, $x - 2y + z = 0$ & $x - y = 0$

Then let $x = \mu$ (for example), so that $y = \mu$ & $z = \mu$, and we have the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, which just represents the usual invariant line through the Origin.

(ii) We want the characteristic equation to be

$(\lambda - 1)^2(\lambda - 2) = 0$

leading to $\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$ (1)

We could then set up a suitably general characteristic equation such as $\begin{vmatrix} a-\lambda & 0 & c \\ 1 & -\lambda & 0 \\ 1 & b & -\lambda \end{vmatrix} = 0$ (2)

and equate coefficients with (1).

Thus (2) becomes $(a - \lambda)\lambda^2 + c(b + \lambda) = 0$ (3)

[A bit of experimentation is necessary in arriving at the determinant in (2), in order that (3) is sufficiently general.]

Then, equating the coefficients in (1) & (3) gives:

$a = 4, c = -5$ & $b = -\frac{2}{5}$

so that (2) becomes $\begin{vmatrix} 4-\lambda & 0 & -5 \\ 1 & -\lambda & 0 \\ 1 & -\frac{2}{5} & -\lambda \end{vmatrix} = 0$

[It is probably worth checking that this produces the required eigenvalues of 1 and 2.]

The eigenvectors corresponding to $\lambda = 1$ are then found from:

$$\begin{pmatrix} 3 & 0 & -5 \\ 1 & -1 & 0 \\ 1 & -\frac{2}{5} & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that $3x - 5z = 0$, $x - y = 0$ & $x - \frac{2y}{5} - z = 0$

Then let $x = \mu$, so that $y = \mu$ & $z = \frac{3\mu}{5}$

Thus an eigenvector is $\begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}$, and we don't have the plane that was found in (i); only a line of invariant points (through the Origin).

[The (advanced) theory behind this is based on the following theorem: "The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity." The algebraic multiplicity is the number of times that the eigenvalue appears as a root of the characteristic equation. The geometric multiplicity is the dimension of the line or plane relating to the eigenvalue: so an invariant line means a geometric multiplicity of 1, whilst an invariant plane means a geometric multiplicity of 2. So, by this theorem, there have to be repeated eigenvalues in order for there to be an invariant plane, but if there are repeated eigenvalues it doesn't follow that there will be an invariant plane.]