Matrices – Q56: Transformations (9/3/24)

For a 2×2 matrix $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with zero determinant:

(i) Find the equation of the line (which we will call the image line) to which all points are mapped.

(ii) Find the family of lines that map to specific points on the image line.

(iii) What condition must apply to a, b, c & d in order for the lines in this family to be perpendicular to the image line?

(iv) Show that if the lines in this family are invariant lines under the transformation, then $tr(M) = 1$, where $tr(M)$ (the trace of M) is defined to be $a + d$

(v) Find the condition(s) for the lines in this family to be invariant lines that are perpendicular to the image line, and give an example of a matrix satisfying these conditions.

Solution

(i) Consider $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

Then $ap + cq = u,\ bp + dq = v$

And, as the determinant is zero, $\frac{b}{a} = \frac{d}{c}$, so that $v = \frac{b}{a}u$;

ie the image line is $y = \frac{b}{a}x$

(ii) Consider a point on the image line with position vector $\begin{pmatrix} u \\ \frac{b}{a}u \end{pmatrix}$.

Then $ap + cq = u$

[$bp + dq = \frac{b}{a}u$ provides no further information]

so that $q = \frac{1}{c}(u - ap)$;

ie the family of lines is $y = -\frac{a}{c}x + \frac{u}{c}$

(iii) The lines in this family will be perpendicular to the image line

when $\left(-\frac{a}{c}\right).\left(\frac{b}{a}\right) = -1$, so that $b = c$

[$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ will then have the form $\begin{pmatrix} a & b \\ b & \frac{b^2}{a} \end{pmatrix}$]

(iv) In order for the lines in this family to be invariant lines, the

point $\begin{pmatrix} u \\ \frac{b}{a}u \end{pmatrix}$ has to lie on the original line (in the family);

ie $\frac{b}{a}u = -\frac{a}{c}u + \frac{u}{c}$

In order for this to hold for all u, $\frac{b}{a} = -\frac{a}{c} + \frac{1}{c}$

and $\frac{bc}{a} = 1 - a$

Then, as $|M| = 0 \Rightarrow \frac{b}{a} = \frac{d}{c}$, it follows that $d = 1 - a$,

and so $tr(M) = 1$, as required.

[In this situation, the image line will be a line of invariant points, and it can be shown that the general requirement for the existence of a line of invariant points is that $tr(M) = |M| + 1$.]

(v) Combining the conditions from (iii) and (iv),

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ has the form $\begin{pmatrix} a & b \\ b & \frac{b^2}{a} \end{pmatrix}$ with $tr(M) = 1$, so that

$a + \frac{b^2}{a} = 1$, and hence $a^2 + b^2 = a$; $b^2 = a(1-a)$,

so that $b = \pm\sqrt{a(1-a)}$, with $a(1-a) \geq 0$; ie $0 \leq a \leq 1$

In conclusion:

$0 \leq a \leq 1$

$c = b = \pm\sqrt{a(1-a)}$

$d = 1 - a$

For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

[Projections of points onto the x and y axes respectively.]

or $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

[$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\binom{p}{q} = \begin{pmatrix} \frac{1}{2}(p+q) \\ \frac{1}{2}(p+q) \end{pmatrix}$, so that the image line (and line of invariant points) is $y = x$; and a member of the family of lines passes through $\binom{p}{q}$ and $\begin{pmatrix} \frac{1}{2}(p+q) \\ \frac{1}{2}(p+q) \end{pmatrix}$, so that its gradient is

$\frac{\frac{1}{2}(p+q)-q}{\frac{1}{2}(p+q)-p} = \frac{\frac{1}{2}(p-q)}{\frac{1}{2}(q-p)} = -1$, as expected.]