Matrices – Invariant Points & Lines

Q53 (9/3/24)

The 2×2 matrix M represents a transformation that maps all points to the same line. In addition, points on this line are invariant points under the transformation. What conditions must apply to M?

Solution

As all points are mapped to the same line, $|M| = 0$.

Let $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, so that $|M| = ad - bc = 0$.

Let $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$

Then $ap + cq = u$ & $bp + dq = v$,

and as $\frac{b}{a} = \frac{d}{c}$ (from $ad - bc = 0$), it follows that $\frac{v}{u} = \frac{b}{a}$ also,

and the equation of the line to which all points map is $y = \frac{b}{a} \, .x$

It is given that all points on this line are invariant:

hence $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} p \\ \frac{b}{a} \, .p \end{pmatrix} = \begin{pmatrix} p \\ \frac{b}{a} \, .p \end{pmatrix}$,

so that $ap + c.\frac{b}{a} \, .p = p$, or $a + c.\frac{d}{c} = 1$

(as this must hold for all p)

ie $a + d$ (the trace of M) $= 1$

[The other equation also gives $bp + d.\frac{b}{a} \, .p = \frac{b}{a}.p \Rightarrow a + d = 1$]

Thus the required conditions are that $|M| = 0$ and $tr(M) = 1$.