Matrices – Q52: Invariant Points & Lines (9/3/24)

For the transformation $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}$:

(i) For what value of k is there a line of invariant points, and find this line.

(ii) Given now that $k = 4$, find any invariant lines of the transformation.

Solution

(i) Suppose that $\begin{pmatrix} 3 & 1 \\ 2 & k \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Then $3p + q = p$ and $2p + kq = q$

so that $q = -2p$ & $2p = q(1 - k)$

Hence $\frac{q}{p} = -2$ & $\frac{q}{p} = \frac{2}{1-k}$

so that $-2 = \frac{2}{1-k} \Rightarrow 1 - k = -1$ & hence $k = 2$

As $q = -2p$, the invariant points lie on the line $y = -2x$

[Check: $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$]

(ii) Suppose that there is an invariant line $y = mx + c$.

Considering a point on this line:

$$\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} 3x + mx + c \\ 2x + 4(mx + c) \end{pmatrix}$$

And, for the line to be invariant,

$2x + 4(mx + c) = m(3x + mx + c) + c$ for all x

Equating coefficients of x: $2 + 4m = 3m + m^2$;

ie $m^2 - m - 2 = 0$

so that $(m - 2)(m + 1) = 0$, and hence $m = 2$ or -1

Equating the constant term: $4c = mc + c$;

ie $c = 0$ or $m = 3$

Combining these two conditions, $m = 2$ or -1, and $c = 0$

So the invariant lines are $y = 2x$ and $y = -x$