Matrices – Q47: Eigenvectors [Practice/M] (8/6/21)

For a reflection in the line $y = x$:

(i) Find the transformation matrix.

(ii) Use eigenvectors to find the invariant lines through the Origin.

(iii) By drawing a diagram, are there any invariant lines that don't pass through the Origin?

(iv) Are there any lines of invariant points?

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Solution

(i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) Characteristic equation is $\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

[Check: trace of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0 + 0 = \lambda_1 + \lambda_2$ & $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 = \lambda_1 \lambda_2$]

For $\boldsymbol{\lambda} = \mathbf{1}$:

$\begin{pmatrix} 0-1 & 1 \\ 1 & 0-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so that $-x + y = 0$, giving an eigenvector of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\boldsymbol{\lambda} = -\mathbf{1}$:

$\begin{pmatrix} 0+1 & 1 \\ 1 & 0+1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

so that $x + y = 0$, giving an eigenvector of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Thus the invariant lines through the Origin are $y = x$ & $y = -x$

(iii) Any line parallel to $y = -x$ is an invariant line. So any line of the form $y = -x + c$, where $c \neq 0$, is an invariant line that doesn't pass through the Origin.

(iv) Invariant points satisfy $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ and are therefore multiples of the eigenvector with eigenvalue 1.

So $y = x$ is the (only) line of invariant points.