Matrices – Q46: Invariant Points & Lines [Practice/M]
(8/6/21)

For the matrix $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, find all of the invariant lines of the associated transformation.

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Solution

Suppose that $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx + c \end{pmatrix} = \begin{pmatrix} x' \\ mx' + c \end{pmatrix}$ for all x,

Then $2x + 2mx + 2c = x'$ & $x + 3mx + 3c = mx' + c$

and $(2 + 2m)x + 2c = x'$ & $(1 + 3m)x + 2c = mx'$

Multiplying the 1st equation by m and equating the two expressions for mx' gives:

$m(2 + 2m)x + 2mc = (1 + 3m)x + 2c$

As this is to hold for all x, we can equate the coefficients of x , to give:

$m(2 + 2m) = 1 + 3m$ & $2mc = 2c$ (1)

Case 1: $c = 0$ (invariant lines through the Origin; ie the eigenvectors)

$2m^2 - m - 1 = 0 \Rightarrow (2m + 1)(m - 1) = 0,$

so that $m = -\frac{1}{2}$ & 1

[Check: To find the eigenvalues of the matrix, we create the characteristic equation $\begin{vmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$\Rightarrow (2 - \lambda)(3 - \lambda) - 2 = 0$

and $\lambda^2 - 5\lambda + 4 = 0$, so that $\lambda_1 = 1$ & $\lambda_2 = 4$

Then, to find the eigenvectors:

$$\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that $x + 2y = 0$; ie one invariant line is $y = -\frac{1}{2}x$

and $\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

so that $x - y = 0$; ie another invariant line is $y = x$

(and these agree with the values of m above)]

Case 2: $c \neq 0$

From (1), $m = 1$ (satisfying both of the equations, in agreement with Exercise (6))

Thus, the other invariant lines are those of the form $y = x + c$

[Also, from Exercise (6), we expect $m = \frac{d-1}{c} = \frac{3-1}{2} = 1$]