Matrices – Q43: Invariant Points & Lines [Problem/H]
(7/6/21)

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Solution

Suppose that $y = px + q$ is a line of invariant points for the transformation $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Then $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} x \\ px + q \end{pmatrix} = \begin{pmatrix} x \\ px + q \end{pmatrix}$

$\Rightarrow ax + cpx + cq = x$

and $bx + dpx + dq = px + q$

As these equations are to hold for any x, we can equate coefficients of x and the constant terms to give:

$a + cp = 1$ (1)

$cq = 0$ (2)

$b + dp = p$ (3)

$dq = q$ (4)

Suppose that $q \neq 0$

Then (2) & (4) $\Rightarrow c = 0$ & $d = 1$

and (1) & (3) $\Rightarrow a = 1$ & $b = 0$

ie $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in which case all points are invariant.

Thus, apart from this trivial case, $q = 0$ and so a line of invariant points takes the form $y = px$, and therefore passes through the Origin.