Matrices – Q4: Eigenvectors [Practice/M](2/6/21)

Find $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3$, using eigenvectors.

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Solution

To find the eigenvalues of $M = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$:

[We want a non-zero solution of $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$;

ie of $(M - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} = 0$; for there to be more than one solution (ie a non-zero solution, as well as the zero solution), $|M - \lambda I| = 0$;
ie $\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$]

The characteristic equation is $\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$,

so that $(2-\lambda)(3-\lambda) - 2 = 0$ and $\lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 4) = 0$

Thus the eigenvalues are $\lambda = 1$ and 4

The eigenvectors satisfy $\begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

For $\lambda = 1$: $x + y = 0 \,; 2x + 2y = 0$

[as a check, these equations should be equivalent, and so producing more than one solution]

Thus an eigenvector for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

For $\lambda = 4$: $-2x + y = 0 \,; 2x - y = 0$

Thus an eigenvector for $\lambda = 4$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

[Note: any multiples of these eigenvectors are also solutions, and so there is an infinite number of solutions]

Let $S = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Then $MS = SD$

[considering the columns of S separately, and noting that $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and so $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; thus $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and similarly for the 1st column]

and hence $M = SDS^{-1}$

so that $M^3 = (SDS^{-1})(SDS^{-1})(SDS^{-1}) = SD^3S^{-1}$

$= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 64 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 1 & 64 \\ -1 & 128 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 66 & 63 \\ 126 & 129 \end{pmatrix} = \begin{pmatrix} 22 & 21 \\ 42 & 43 \end{pmatrix}$

[we would obviously expect to have only integers in the answer, being a power of M]

Check: $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}^3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 10 & 11 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 22 & 21 \\ 42 & 43 \end{pmatrix}$