Mappings & Functions (3 pages; 2/6/23)

(1) Mappings

Consider the following examples of mappings, where $x \in \mathbb{R}$

(a) $f(x) = 2x$

This is a $1 - 1$ mapping. The lefthand 1 refers to the fact that eg only 1 value of x maps to 4 (namely 2), whilst the righthand 1 refers to the fact that eg there is only one image of 3 (namely 6).

(b) $g(x) = x^2$

This is a $many - 1$ mapping. The lefthand 'many' refers to the fact that eg there is more than 1 value of x that maps to 9 (namely 3 & -3). As for (a), there is eg only one image of 4 (namely 16); hence the righthand 1.

Any mapping with a righthand 1 is described as a function.

An injective function (or injection) is a 1-1 function (ie $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$).

(c) $h(x) = \pm\sqrt{x}$

This is a $1 - many$ mapping. The lefthand 1 refers to the fact that eg there is only 1 value of x that maps to 5 (namely 25). As eg there is more than 1 image of 49 (namely $7 \& -7$), we have a righthand 'many', and $h(x)$ is therefore not a function.

(d) $j(x) = \pm\sqrt{25 - x^2}$ (a circle)

This is a $many - many$ mapping. The lefthand 'many' refers to the fact that eg there is more than 1 value of x that maps to 3 (namely $4 \& -4$). And as eg there is more than 1 image of 0 (namely $5 \& -5$), we have a righthand 'many', and $j(x)$ is therefore not a function.

(2) Domain, Codomain, Image (& Range)

A function $y = f(x)$ can be defined as a mapping from the set of values X to the set of values Y, such that there is only one possible y for each x. The domain of the function is X, and the codomain is Y (ie they are both part of the definition of the function).

For every x in the domain, there must be a $y[= f(x)]$ in the codomain; ie the domain consists only of values for which the function is defined.

Suppose that a function is defined in the following way:

$f: \mathbb{Z} \to \mathbb{Z}; f(x) = 2x$ [ie specifying that $x \in \mathbb{Z}$ & $f(x) \in \mathbb{Z}$],

then the domain of f is defined as the set of allowable values for x; ie $\mathbb{Z}$ in this case.

The codomain is defined as the set of allowable values for $f(x)$;

ie also $\mathbb{Z}$ in this case [note that this is before we have investigated whether the nature of the function places any constraints on the possible values of $f(x)$].

The image of f is defined as the set of values that $f(x)$ will actually take; ie in this case allowing for the fact that $2x$ has to be even; so the image here is the set of even integers.

Where the codomain is $\mathbb{R}$ (or a subset of $\mathbb{R}$), then the image is usually the same as the codomain.

If the image is the same as the codomain, then the function is said to be surjective. Otherwise the image will be a proper subset of the codomain.

A bijective function (or bijection) is a function that is both injective and surjective; in other words, a $1-1$ function where the image is the same as the codomain. For example, $f: \mathbb{R} \to \mathbb{R}; f(x) = 2x$.

Unfortunately, there is no universal agreement as to the definition of the range of a function. In MEI textbooks, it is taken to be the same thing as the image, but sometimes it is taken to be the same thing as the codomain. It is safest to use the term image instead (if that is what is intended).

(3) Further Examples

(i) $y = x + 2$; domain: $x \in \mathcal{R}, x > 0$; codomain : $y \in \mathcal{R}, y > 2$

Then the image is $y \in \mathcal{R}, y > 2$ (the same as the codomain).

[Note that a codomain of $y \in \mathcal{R}, y > 3$ (for example) wouldn't be consistent with a domain of $x \in \mathcal{R}, x > 0$, as it wouldn't include the image of $x = 1$; ie $y = 1 + 2 = 3$.]

(ii) $y = 2x$; domain: $x \in \mathbb{Z}$; codomain : $y \in \mathbb{Z}$

Then the image is $y \in \mathbb{Z}$, y is even. [Note that a negative number can be odd or even, and that zero is even.]