Maclaurin Series (7 pages; 25/3/18)

See also "Taylor Series"

(1) Suppose that the function $f(x)$ can be represented by a polynomial of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

Then $f(0) = a_0$

$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \ldots$ and $f'(0) = a_1$

$f''(x) = 2a_2 + (3)(2)a_3x + (4)(3)a_4x^2 \ldots$ and $f''(0) = 2a_2$

$f'''(x) = (3!)a_3 + (4!)a_4x + \ldots$ and $f'''(0) = (3!)a_3$

Thus, in general, $a_r = \frac{f^{(r)}(0)}{r!}$

and $f(x) = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!} x^r$

This is the Maclaurin expansion of $f(x)$. This is sometimes referred to as Maclaurin's theorem.

(2) The Maclaurin expansion of $f(x)$ to the nth power is also called the Maclaurin polynomial for $f(x)$ of degree n; sometimes denoted by $p_n(x)$.

Generally speaking, the expressions ‘Maclaurin series’ and ‘Maclaurin expansion’ are used interchangeably to refer to an infinite series.

(3) **Example**: $f(x) = (1-x)^{-1}$

$f(0) = 1$

$f'(x) = (1-x)^{-2}$, so that $f'(0) = 1$

Then $f''(x) = 2(1-x)^{-3}$, so that $f''(0) = 2!$

and $f'''(x) = 3!\,(1-x)^{-4}$, so that $f'''(0) = 3!$

Hence $f(x) = 1 + x + x^2 + x^3 + \cdots$

(A geometric series with sum to infinity of $\frac{1}{1-x}$ when $|x| < 1$;

the range of validity of a Maclaurin series will be discussed shortly.)

(4) **Example**: $f(x) = cosx$ (x in radians)

$f(0) = 1$

Alternate derivatives will be $\pm sinx$, so that $f^{(r)}(0) = 0$ when r is odd.

There is a sign change whenever $cosx$ is differentiated.

So $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Similarly for $sinx$: even powers vanish.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(using the fact that $sinx \approx x$ for small x)

(5) **Example**: $f(x) = (1+x)^n$, where n is any non-zero real number.

$f(0) = 1$

$f'(x) = n(1+x)^{n-1}$ and $f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2}$ and $f''(0) = n(n-1)$

Hence $f(x) = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

(6) Range of validity

Some expansions are valid for all x; such as the following:

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(ii) $sinx\ \ and\ \ cosx$

(iii) $(1+x)^n$, where n is a positive integer

Note: the techniques used to establish the range (or 'interval') of validity are outside the A Level syllabus.

In order for an expansion to be valid, the derivatives of $f(x)$ must all exist at $x = 0$, as well as the function itself.

Example 1: If $f(x) = lnx$, then $f(0)$ isn't defined, and hence there isn't an expansion for lnx. However, one can be found for $\ln(x+1)$ instead.

Example 2: If $f(x) = |x|$, then $f'(0)$ is undefined.

Example 3: If $f(x) = e^{\sqrt{x}}$, then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^{\sqrt{x}}$ so that $f'(0)$ isn't defined. A Taylor series expansion, centred on a value other than 0 could be found instead. [A Taylor series is a more general version of the Maclaurin series.]

(7) Expansions with limited validity

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$,

where $n \notin \mathbb{Z}^+$, for $-1 < x < 1$

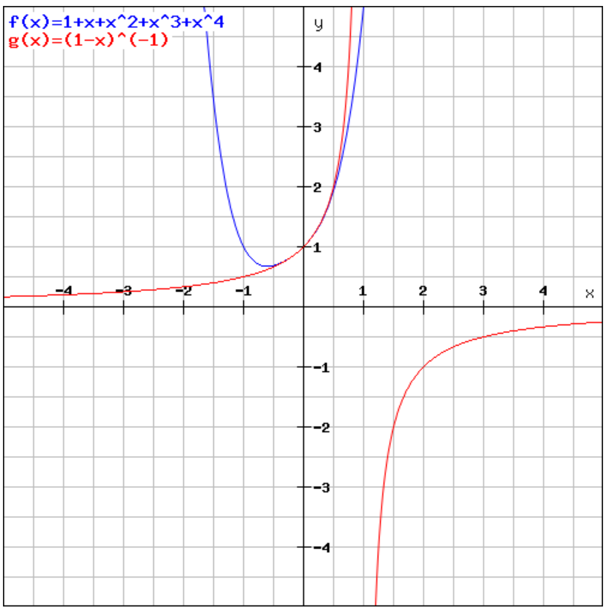
(ii) $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$ for $-1 < x \le 1$

[Note that $y = \ln(1+x)$ passes through the Origin, and has a gradient of $\frac{1}{1+x}$, which equals 1 at the Origin; so that

$\ln(1+x) \approx x$ is a 1st approximation. Also the gradient is seen to reduce with increasing x, so that we would expect the next term of the series to be negative.]

(iii) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \leq x \leq 1$

(8) Accuracy of approximations



(i) **Example**

$f(x) = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$

$1 + x + x^2$ is close to $f(x)$ in a small interval around $x = 0$, and successive approximations get closer to $f(x)$ for increasingly wider intervals of x

(ii) It is often possible to come up with indirect approximations, which use values of x closer to 0 (about which the Maclaurin expansion is centred), so that greater accuracy is achieved.

[See for example Ex. 6 in "Maclaurin Series - Exercises"]

(iii) If an approximation is required for a value that isn't that close to 0, then a Taylor series expansion, centred about the value in question, is likely to be more appropriate.

(9) **Example**: Find $\sin\frac{\pi}{3}$ to 3dp , using the Maclaurin expansion of $sinx$.

Solution

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ ...$

Successive approximations:

1.04720, 0.85580, 0.86630, 0.86602 $[\frac{\sqrt{3}}{2} = 0.86603\ (5sf)]$

(10) Options for dealing with more complicated functions

(i) eg $f(x) = e^{x^2}$

As an alternative to obtaining the Maclaurin expansion directly (ie by finding $f'(x)$, $f''(x)$ etc), simply substitute x^2 for x in the expansion for e^x, so that $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$ for $x \in \mathbb{R}$

(ii) Some functions will need to be rearranged, so as to fit the standard form necessary for the Maclaurin expansion to be applied.

(a) eg $f(x) = (2+x)^{-1/2} = 2^{-\frac{1}{2}}(1+\frac{x}{2})^{-1/2}$ etc

(b) eg $f(x) = \ln(4x) = ln4 + lnx = ln4 + \ln(1 + [x-1])$ etc

(iii) eg $f(x) = sinxe^x$

If an expansion is only required up to a particular power of x, then we can just write:

$f(x) = (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots)(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots),$

and collect up the required powers.

Example: Find the Maclaurin expansion of sin^2x, as far as the term in x^4

Solution

$sin^2x = (x - \frac{x^3}{3!} + \cdots)(x - \frac{x^3}{3!} + \cdots)$

$= x^2 - \frac{2x^4}{3!} + \ldots$

$= x^2 - \frac{x^4}{3} + \ldots$

(iv) If a Maclaurin expansion can be found for $f'(x)$, then the Maclaurin expansion for $f(x)$ can be obtained by integrating the expansion of $f'(x)$

eg the Maclaurin expansion for $sinx$ could be obtained by integrating the expansion for $cosx$:

$$\int 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \dots \ dx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \dots$$

(with $c = 0$)

[the interval of validity of $f(x)$ can be shown to be at least as wide as that of $f'(x)$]

(v) eg $f(x) = \frac{x}{\ln(1+x)}$

An 'ad-hoc' approach in this case is as follows:

$$f(x) = \frac{x}{x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots} = \frac{1}{1 - \frac{x}{2} + \frac{x^2}{3} - \cdots}$$

$$= (1-u)^{-1} \quad \text{where } u = \frac{x}{2} - \frac{x^2}{3} + \cdots$$

$$= 1 + \left(\frac{x}{2} - \frac{x^2}{3} + \cdots\right) + \left(\frac{x}{2} - \frac{x^2}{3} + \cdots\right)^2 + \cdots,$$

which can be expanded to the required number of terms.

However, for these composite functions the interval of validity will not be known. The Maclaurin polynomials just give good approximations to $f(x)$ when x is small.

(vi) If $f(x) = e^x cosx$, then $f'(x) = e^x(cosx - sinx)$

and $f''(x) = f'(x) - e^x(sinx + cosx) = f'(x) + f'(x) - 2f(x)$

$= 2(f'(x) - f(x))$; so that $f^{(3)}(x) = 2(f''(x) - f'(x))$ etc