Maclaurin Series – Q6 [Practice/M] (2/6/21)

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Solution

Let $f(x) = cosh^3x \qquad [f(0) = 1]$

Then $f'(x) = 3cosh^2xsinhx \quad [f'(0) = 0]$

and $f''(x) = 6coshxsinh^2x + 3cosh^3x \quad [f''(0) = 3]$

$= 6coshx(cosh^2x - 1) + 3cosh^3x = 9cosh^3x - 6coshx$

$= 9f(x) - 6coshx$

Then $f'''(x) = 9f'(x) - 6sinhx \quad [f'''(0) = 0]$

and $f^{(4)}(x) = 9f''(x) - 6coshx$

$= 81f(x) - 6coshx(1 + 9) \quad [f^{(4)}(0) = 81 - 60 = 21]$

Clearly $f^{(2r+1)}(0) = 0$ and hence $a_{2r+1} = 0$

$f^{(6)}(x) = 81f''(x) - 6coshx(1 + 9)$

$= 81(9)f(x) - 6coshx(1 + 9 + 81)$

Thus $f^{(2r)}(x) = 3^{2r}f(x) - 6coshx(1 + 9 + \cdots + 9^{r-1})$

and so $a_{2r} = \frac{f^{(2r)}(0)}{(2r)!} = \frac{1}{(2r)!}\{3^{2r} - 6\frac{(9^r-1)}{(9-1)}\}$

$= \frac{1}{(2r)!}\left\{3^{2r} - 3\frac{(3^{2r}-1)}{4}\right\}$

$= \frac{3^{2r}+3}{4(2r)!}$

So $cosh^3x = 1 + \frac{3x^2}{2!} + \frac{21x^4}{4!} + \cdots + \frac{(3^{2r}+3)x^{2r}}{4(2r)!} + \cdots$

$$= 1 + \frac{3x^2}{2} + \frac{7x^4}{8} + \cdots + \frac{(3^{2r}+3)x^{2r}}{4(2r)!} + \cdots$$

Note: In other cases, such as $cosh^2x$, if only required to find the first few terms of the expansion, it may well be easier to expand

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!}\right)$$