MAT Problems - Miscellaneous (Sol'ns) (11 pages; 19/9/17)

(1) Consider the quadratic equation $x^2 + bx + c = 0$

(i) By experimenting with different examples, find conditions on $b\ and/or\ c$ for the roots of the equation to exist and be of the same sign.

(ii) Find conditions for the roots to exist and both be positive

Solution

(i) First of all, $b^2 - 4c \geq 0$ is needed, in order for there to be real roots.

Then $c > 0$ will ensure that the roots are either both positive or both negative; whilst $c < 0$ ensures that the roots are of different sign (see Fig. 1 below). Clearly $c = 0$ gives one zero root.

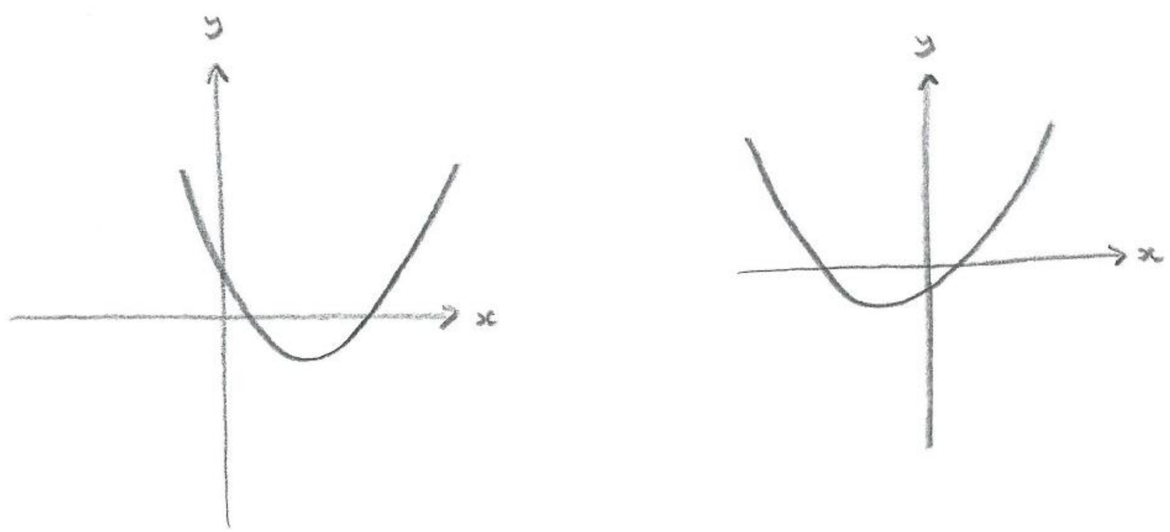


Fig. 1

So we want $b^2 \geq 4c$ & $c > 0$

or $c > 0$ & $|b| \geq 2\sqrt{c}$ (as an alternative form)

(ii) In order for the roots to be both positive, we require the above conditions to apply, and in addition for the minimum to have a positive x coordinate.

As the minimum is halfway between the two roots (assuming they exist), and in this example has x coordinate $-\frac{b}{2}$ (from the quadratic formula; in general it is $-\frac{b}{2a}$), we therefore want $b < 0$.

So our conditions become:

$c > 0$ & $|b| \geq 2\sqrt{c}$ & $b < 0$

which simplifies to

$c > 0$ & $b < -2\sqrt{c}$

Alternative Approach

We could instead require the smaller of the two roots to be positive, so that $-b - \sqrt{b^2 - 4c} > 0$

$\Leftrightarrow -b > \sqrt{b^2 - 4c}$

$\Leftrightarrow b < 0$ & $|b| > \sqrt{b^2 - 4c}$

$\Leftrightarrow b < 0$ & $c > 0$, together with $b^2 - 4c \geq 0$ (in order for the roots to exist), as before

(2) Find the square roots of $49 - 12\sqrt{5}$

Solution

Let $x^2 = 49 - 12\sqrt{5}$

Consider $x = a + b\sqrt{5}$

Then $a^2 + 2ab\sqrt{5} + 5b^2 = 49 - 12\sqrt{5}$

Let $a^2 + 5b^2 = 49$ & $2ab = -12$

[a variation on Equating Coefficients]

Looking for integer solutions, we see that either

$a = 2$ & $b = -3$ or $a = -2$ & $b = 3$ work.

(3) Consider the sequence defined by $u_n = au_{n-1} + b$,

where a & b are real constants, and u_0 is given.

(i) What familiar sequences are special cases of this sequence?

Solution

Setting $a = 1$ gives an arithmetic sequence.

Setting $b = 0$ gives a geometric sequence.

(ii) Define a new sequence by $v_n = u_n + c$

For what value of c, in terms of a & b, will v_n be a geometric sequence?

For what value of a does this not work?

Solution

$v_{n-1} = u_{n-1} + c$, and hence

$u_n = au_{n-1} + b \Rightarrow v_n - c = a(v_{n-1} - c) + b$

$\Rightarrow v_n = av_{n-1} + b + c(1 - a)$

For v_n to be a geometric sequence, we want $b + c(1 - a) = 0$, so that $c = \frac{-b}{1-a} = \frac{b}{a-1}$, provided that $a \neq 1$

When $a = 1$, u_n, and hence v_n also, are arithmetic sequences.

(iii) If $u_n = 2u_{n-1} + 3$, and $u_0 = 4$, find a formula for u_n in terms of n

Solution

From (ii), $c = \frac{3}{2-1} = 3$ and $v_n = 2v_{n-1}$

Then $v_n = v_0(2^n)$

and $v_n = u_n + 3$, so that $u_n + 3 = (u_0 + 3)(2^n)$

and $\therefore u_n = 7(2^n) - 3$

(and this can be checked by comparing with $u_n = 2u_{n-1} + 3$, and $u_0 = 4$)

(iv) Find a similar formula for $u_n = au_{n-1} + b$, where u_0 is given.

Solution

From (ii), $c = \frac{b}{a-1}$ and $v_n = av_{n-1}$

Then $v_n = v_0(a^n)$

and $v_n = u_n + c$, so that $u_n + c = (u_0 + c)(a^n)$

and $\therefore u_n = (u_0 + c)(a^n) - c = \left(u_0 + \frac{b}{a-1}\right)(a^n) - \frac{b}{a-1}$

(v) Under what conditions will u_n be constant? Give a non-trivial example.

Solution

Either $a = 1$ & $b = 0$

Or $a = 0$ and $u_0 = b$

Or $u_0 + \frac{b}{a-1} = 0$; ie $u_0 = \frac{b}{1-a}$

For example, $u_n = 2u_{n-1} - 1$, where $u_0 = 1$

(4) Find the turning points of $y = (x^2 - 4x + 3)^2$

Solution

Method 1

As $x^2 - 4x + 3 = (x-1)(x-3)$,

$y = (x-1)^2(x-3)^2$

Then $\frac{dy}{dx} = 2(x-1)(x-3)^2 + (x-1)^2(2)(x-3)$

$= 2(x-1)(x-3)(x-3+x-1)$

$= 4(x-1)(x-3)(x-2)$

$\frac{dy}{dx} = 0$ when $x = 1, 2$ & 3

At $x = 1$, $\frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

At $x = 2$, $\frac{dy}{dx}$ changes from +ve to -ve, indicating a max. point.

At $x = 3$, $\frac{dy}{dx}$ changes from –ve to +ve, indicating a min. point.

The min. points are therefore (1, 0) and (3, 0), whilst the max. is at (2,1).

Method 2

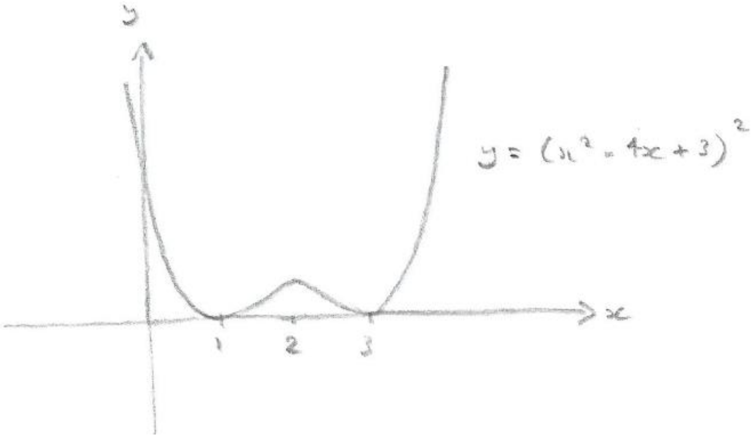
$(x^2 - 4x + 3)^2 \geq 0$ and $(x^2 - 4x + 3)^2 = (x-1)^2(x-3)^2 = 0$ has roots at $x = 1$ & 3, so that there are minima at these two points.

For $x = 1 - t$, $y = x^2 - 4x + 3$ and hence y $= (x^2 - 4x + 3)^2$ increases as t increases, and similarly for $x = 3 + t$.

For $1 < x < 3$, y $= (x^2 - 4x + 3)^2$ attains a max. when

$x^2 - 4x + 3$ (which is negative in this range) is at a min. ; ie when

$x = 2$.



(5) Show that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$

Solution

Method 1: Consider $(1+1)^n$

Method 2: Pascal's triangle

The sum of each row is twice the sum of the previous one.

eg $1 + 5 + 10 + 10 + 5 + 1$

$= (1 + 10 + 5)[alternate\ terms] + (5 + 10 + 1)$

$= 2(1 + 10 + 5) = 2(1 + [4 + 6] + [4 + 1])$

& $1 + 6 + 15 + 20 + 15 + 6 + 1$

$= (1 + 15 + 15 + 1) + (6 + 20 + 6)$

$= (1 + [5 + 10] + [10 + 5] + 1)$

$+([1 + 5] + [10 + 10] + [5 + 1])$

Method 3: Counting ways of selecting any number of items

1st counting method: $\sum_{r=0}^{n} \binom{n}{r}$

2nd counting method: For each object, there are 2 choices: include or exclude; giving 2^n

[Note: 1 way of choosing no objects is included in the total.]

Method 4: Induction

If true for $n = k$, so that $\sum_{r=0}^{k} \binom{k}{r} = 2^k$,

then $\sum_{r=0}^{k+1} \binom{k+1}{r} = \binom{k+1}{0} + \{\sum_{r=1}^{k} \binom{k+1}{r}\} + \binom{k+1}{k+1}$

$= 1 + \sum_{r=1}^{k} \left\{\binom{k}{r-1} + \binom{k}{r}\right\} + 1$

$= 1 + \{\sum_{r-1=0}^{k-1} \binom{k}{r-1}\} + [\{\sum_{r=0}^{k} \binom{k}{r}\} - \binom{k}{0}] + 1$

$= 1 + \{\sum_{R=0}^{k-1} \binom{k}{R}\} + [2^k - 1] + 1$

$= 1 + \{\sum_{R=0}^{k} \binom{k}{R}\} - \binom{k}{k} + 2^k$

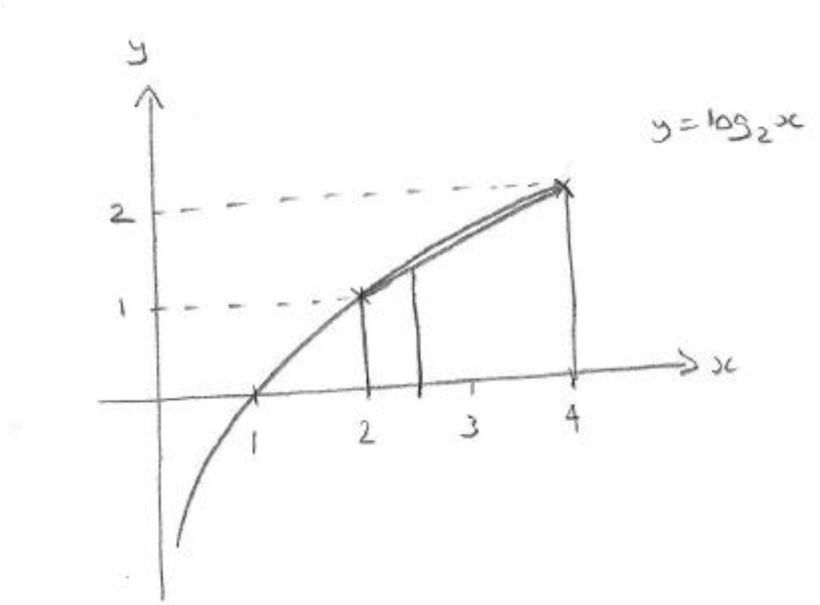
$= 1 + 2^k - 1 + 2^k$

$= 2^{k+1}$

(6) Linear Interpolation

By approximating the graph of $y = log_2 x$ by a straight line between $x = 2$ and $x = 4$, find an approximate value for $log_2 \left(\frac{5}{2}\right)$

Solution



Approach 1: weighted average

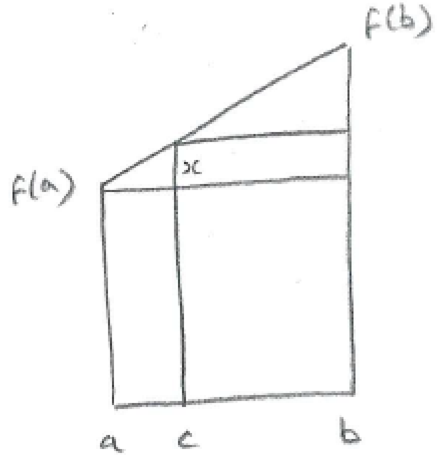
$log_2\left(\frac{5}{2}\right) \approx \left(\frac{4-2.5}{4-2}\right) log_2 2 + \left(\frac{2.5-2}{4-2}\right) log_2 4$

$= (0.75)(1) + (0.25)(2) = 1.25$

Approach 2: similar triangles

Referring to the diagram below (for the general function $f(x)$)

$\frac{x}{c-a} = \frac{f(b)-f(a)}{b-a}$



For our example,

$\frac{x}{2.5-2} = \frac{2-1}{4-2}$,

so that $x = (0.5)(0.5) = 0.25$, and hence $log_2\left(\frac{5}{2}\right) \approx 1 + 0.25 = 1.25$

Approach 3: Equation of line

The gradient of the line is $\left(\frac{f(b)-f(a)}{b-a}\right)$

Then $f(c) \approx f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(c-a)$

In this case, $log_2\left(\frac{5}{2}\right) \approx 1 + \left(\frac{2-1}{4-2}\right)(2.5-2) = 1.25$ again.

Also $f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(c-a)$

$= \left(\frac{1}{b-a}\right)\left((b-a)f(a) + (c-a)f(b) - (c-a)f(a)\right)$

$= \left(\frac{1}{b-a}\right)\left((b-c)f(a) + (c-a)f(b)\right)$,

which is the weighted average approach

(7) (i) Does $\sqrt{4}$ equal 2 or ± 2? (ii) Simplify $\sqrt{x^2}$

Solution

(i) By convention, 2 (consider the $\pm$ in the quadratic formula).

(ii) $|x|$

(8) For what value of x does $(x+2)(x+4)$ have its minimum value?

Solution

Roots of $(x+2)(x+4)=0$ are -2 & -4, so minimum is at

$x=-3$ (or complete the square, or find stationary point)

(9) Prove that $E' \Rightarrow L'$ is equivalent to $L \Rightarrow E$

Solution

Suppose that L is true & E is not true; then $E' \Rightarrow L'$ means that L is not true; ie a contradiction; hence $L \Rightarrow E$

(10) Give an example of a quadratic equation that has no real roots.

Solution

Anything of the form $(x+a)^2+b^2=0$ (where a & b are Real numbers, and $b \neq 0$);

eg $(x+1)^2+1=x^2+2x+2=0$

(11) If $\int_{-a}^{a} f(x)\, dx = b$, find $\int_{-a}^{a} f(-x)\, dx$

Solution

b also, as $f(-x)$ is the reflection of $f(x)$ in the y-axis