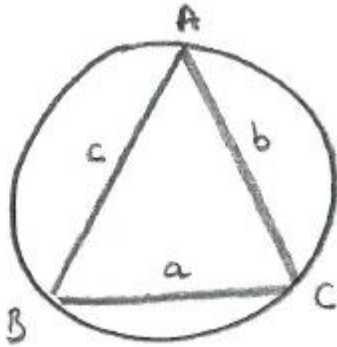


## MAT Problems - Geometry (Sol'ns) (6 pages; 19/9/17)

(1) ABC is a triangle circumscribed by a circle of radius R, as shown in the diagram below.

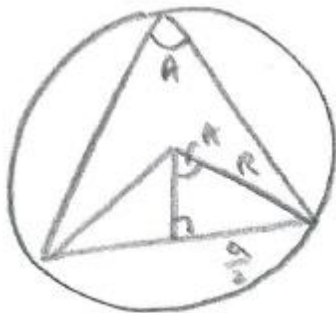


Show that (i)  $\frac{a}{\sin A} = 2R$  (ii) the area of the triangle is  $\frac{abc}{4R}$

### Solution

(i) Drawing radii from B and C to the centre of the circle, as in the diagram below, and noting that the angle at the centre is twice the angle at the circumference,

$\sin A = \frac{\left(\frac{a}{2}\right)}{R}$ , so that  $\frac{a}{\sin A} = 2R$ , as required



(ii) Area of  $ABC = \frac{1}{2}bc\sin A = \frac{1}{2}bc\left(\frac{a}{2R}\right) = \frac{abc}{4R}$

(2) Find the equation of the circle passing through the points

A (2,8) , B (7,3) and D (5,7)

### Solution

The first step is to find the centre of the circle, using the fact that the perpendicular bisector of each chord passes through the centre.

The chord AB has mid-point  $(9/2, 11/2)$

and gradient  $\frac{3-8}{7-2} = -1$

The perpendicular bisector of AB therefore has equation

$$\frac{y-11/2}{x-9/2} = -\frac{1}{-1}$$

$$\rightarrow 2y - 11 = 2x - 9$$

$$\rightarrow y = x + 1$$

The chord BD has mid-point (6, 5)

and gradient  $\frac{7-3}{5-7} = -2$

The perpendicular bisector of BD therefore has equation

$$\frac{y-5}{x-6} = -\frac{1}{-2}$$

$$\rightarrow y = \frac{1}{2}x + 2$$

The centre of the circle C is then found from the intersection of these lines:

$$x + 1 = \frac{1}{2}x + 2$$

so that  $x = 2$  and  $y = 3$

The radius is then the distance CA (for example)

$$= \sqrt{(2 - 2)^2 + (3 - 8)^2} = 5$$

Hence the equation of the circle is  $(x - 2)^2 + (y - 3)^2 = 25$

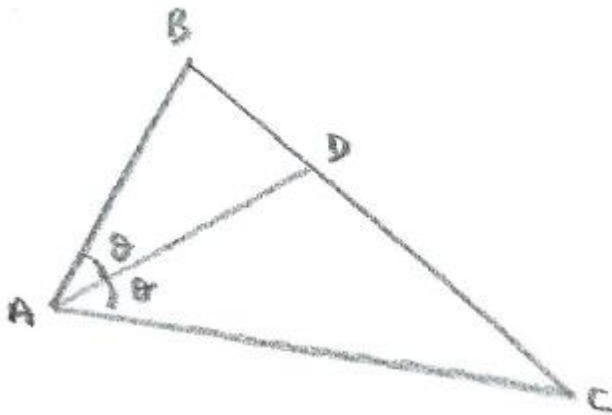
(Check: B and D satisfy the equation.)

### (3) Angle Bisector Theorem

Referring to the diagram below, the Angle Bisector theorem says that

$$\frac{BD}{DC} = \frac{AB}{AC}$$

Prove the Angle Bisector Theorem.



### Solution

By the Sine rule for triangle ABD,  $\frac{BD}{\sin\theta} = \frac{AB}{\sin ADB}$  (1)

and, for triangle ADC,  $\frac{DC}{\sin\theta} = \frac{AC}{\sin ADC} = \frac{AC}{\sin ADB}$  (2)

Then (1)  $\Rightarrow \frac{\sin\theta}{\sin ADB} = \frac{BD}{AB}$  and (2)  $\Rightarrow \frac{\sin\theta}{\sin ADB} = \frac{DC}{AC}$

so that  $\frac{BD}{AB} = \frac{DC}{AC}$

and hence  $\frac{BD}{DC} = \frac{AB}{AC}$

(4) Find the shortest and longest distances from the point (3,4) to the circle  $x^2 + y^2 = 144$  (justifying any assumptions you make).

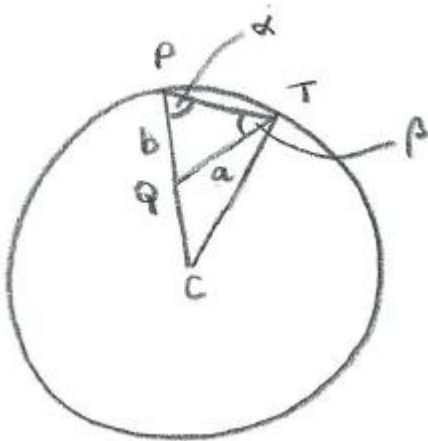
### Solution

#### Approach 1

The nearest point  $P_1$  on the circle to  $Q$  (3,4) will be such that  $P_1Q$  is perpendicular to the tangent at  $P_1$ .

This can be justified as follows:

If  $PQ$  is perpendicular to the tangent at  $P$ , then  $PQ$  extended will pass through the centre of the circle,  $C$  (by a property of the tangent to a circle). In the diagram below, we wish to show that  $a > b$  (where  $a = QT$  &  $b = QP$ ); ie that any neighbouring point,  $T$  on the circle will be further from  $Q$  than  $P$ .



[Note: This diagram and the following one are applicable for any point  $Q$ .]

As  $CPT$  is an isosceles triangle, angles  $CPT$  and  $CTP$  are equal, and hence  $\alpha > \beta$ . Then  $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$  and  $\frac{a}{b} = \frac{\sin\alpha}{\sin\beta} > 1$  (as  $\alpha > \beta$ ,

and  $\sin x$  is an increasing function for  $0 < x < \frac{\pi}{2}$ , so that  $a > b$ , as required.

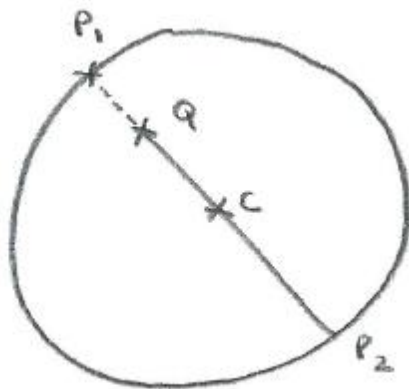
Thus we have shown that  $P_1Q$  is perpendicular to the tangent at  $P_1$ , and hence (as already mentioned)  $Q$  lies on the radius from  $P_1$ . So the distance  $P_1Q = 12 - 5 = 7$ , since  $OQ = 5$  (by Pythagoras).

To find the furthest point from  $Q$ :

Let  $P$  be a point on the circle and  $P_2$  the point furthest from  $Q$ .

The maximum value that  $PQ$  can possibly take is the diameter

less  $P_1Q$ , and this can be achieved if  $P_1, Q, C$  &  $P_2$  lie on the same diameter, as in the diagram below.



In this situation,  $P_2Q = P_2C + CQ = 12 + 5 = 17$

## Approach 2

Let  $(x, y)$  be a point  $P$  on the circle, so that  $x^2 + y^2 = 144$  (A)

The square of the distance from  $(3,4)$  to  $P$  is:

$$D^2 = (x - 3)^2 + (y - 4)^2$$

We wish to find stationary points of  $D^2$ :

$$\frac{d}{dx} D^2 = 0 \Rightarrow 2(x - 3) + 2(y - 4) \frac{dy}{dx} = 0 \quad (\text{B})$$

Then, differentiating (A),  $2x + 2y \cdot \frac{dy}{dx} = 0 \quad (\text{C})$

and (B) & (C) give  $(x - 3) + (y - 4) \left(-\frac{x}{y}\right) = 0,$

so that  $xy - 3y - xy + 4x = 0,$

and hence  $y = \frac{4x}{3}$

This is the equation of the diameter through (4,3), which establishes that  $P_1$ ,  $P_2$  &  $Q$  all lie on a diameter. The lengths  $P_1Q$  and  $P_2Q$  are then calculated as before.