MAT Problems F - Counting (Sol'ns) (5 pages; 19/9/17)

(1) A 4-digit password is made up of numbers from 0 to 4, where the numbers can be repeated, but have to be ordered from largest to smallest. Show that there are 70 possible passwords.

Solution

Consider a simpler version of the problem, with just 3 numbers, each of which can be $0, 1\ or\ 2$

Then the following are possible:

$222, 221, 220, 211, 210, 200$

$111, 110, 100$

000

It may be possible to come up with a recurrence relation of some sort. Let $f(m, n)$ be the number of possibilities when we can choose between $0, 1, \ldots, m$ for each digit, and there are n digits in the password.

From the above, we have $f(2,3) = f(2,2) + f(1,2) + f(0,2)$

and this can be generalised to

$f(m, n) = f(m, n-1) + f(m-1, n-1) + \cdots + f(0, n-1)$

We also note that $f(m, 1) = m + 1$

Applying this to the problem in question,

$f(4,4) = f(4,3) + f(3,3) + f(2,3) + f(1,3) + f(0,3)$

$= [f(4,2) + f(3,2) + \cdots + f(0,2)]$

$+[f(3,2) + f(2,2) + \cdots + f(0,2)]$

$+ \cdots + f(0,2)$

$= f(4,2) + 2f(3,2) + 3f(2,2) + \cdots + 5f(0,2)$

$= [f(4,1) + f(3,1) + \cdots + f(0,1)]$

$+2[f(3,1) + f(2,1) + \cdots + f(0,1)]$

$+ \cdots 5f(0,1)$

$= f(4,1) + (1+2)f(3,1) + (1+2+3)f(2,1)$

$+(1+2+3+4)f(1,1) + (1+2+3+4+5)f(0,1)$

$= 5 + 3(4) + 6(3) + 10(2) + 15(1)$

$= 5 + 12 + 18 + 20 + 15 = 70$

[This method can be applied to the original BMO problem, though it would become unwieldy if the length of the password exceeded 6 digits.]

Alternative (quicker) method [based on the official solutions, contained in "A Mathematical Olympiad Primer" by Geoff Smith]:

Each possibility can be represented using the following system:

3220 is represented by DXDXXDDX

and 4311 is represented by $XDXDDXXD$

For 3220, the 1st letter D means that we are dropping by 1 from the maximum of 4, and the 2nd letter X means that we have reached the value of the 1st digit; similarly the next 2 letters DX mean that we are dropping by another 1 to arrive at the 2nd digit; the 5th letter X means that we don't drop at all to arrive at the 3rd digit; the final letters DDX mean that we drop by 2 to arrive at the last digit.

For 4311, we need a D on the end to get down to 0.

In all cases there will be 4 Ds and 4 Xs, and the Ds and Xs can appear in any of the places, so that the number of possibilities is $\binom{8}{4} = 70$.

[See the official solutions for a couple of other approaches.]

(2) Given 6 pairs of twins, in how many ways can they be placed in 3 teams of 4, such that no team contains any pair of twins?

Solution

Let the 6 pairs of twins be labelled $AaBb \dots Ff$

Define team 1 to be the team containing A.

Team 1 might, for example, be $AbCd$.

The number of ways of choosing the 3 people to go with A is

$\binom{5}{3}$ [the number of ways of choosing 3 of the 5 remaining pairs]

$\times\ 2^3$ [as either twin could be chosen for each pair]

For the case where team 1 is $AbCd$, define team 2 to be the team containing E. [Had E been chosen for team 1, it would have been another letter that was chosen to define team 2]

One complete selection of teams would then be:

team 1: $AbCd$

team 2: $a\ \ c\ \ \ Ef$

team 3:   B DeF

There is the following scope for choice:

$\binom{4}{2}$ ways of choosing the 2 people out of $aBcD$ to go in team 2; combined with the 2 ways of choosing either $f\ or\ F$ for team 2

Thus the overall total number of ways is:

$\binom{5}{3} \times 2^3 \times \binom{4}{2} \times 2 = 10 \times 8 \times 6 \times 2 = 960$

Note: For team 1, an alternative calculation is as follows:

The number of ways of filling the remaining 3 places in team 1 is $10 \times 8 \times 6$, if order is important; but, as order isn't important, we divide by 3!, to give 10×8, as before.

(3) Five poorly-behaved pupils are required to sit in the front five places in a classroom. Angus insists on sitting next to Bruce, Chantal refuses to sit next to Deborah, and Emily is happy to sit anywhere. In how many different ways can they take their seats?

A 24 B 12 C 30 D 20 E 6

Solution

Let X represent A & B.

Suppose that X is to the left of E.

Then C & D can go in the spaces shown here: _X_E_

There are 3 choices for C, and then 2 choices for D, giving 6 possibilities.

Multiply by 2, to include cases where X is to the right of E; giving 12 possibilities.

Multiply by 2 again, as X could be AB or BA, giving 24 possibilities.

Answer is A

Alternative approach

Number of ways with no constraints on C & D, where A is ahead of B (eg XABXX):

4 (ways of placing A)

$\times$ 3! (ways of placing C, D & E)

$= 24$

Including cases where B is ahead of A gives $24 \times 2 = 48$ (1)

Permutations to be excluded, with A ahead of B and C ahead of D:

ABCDE, ABECD, EABCD, CDABE, CDEAB, ECDAB

giving a total of $6 \times 2 \times 2 = 24$ to be excluded (including cases where B is ahead of A and/or D is ahead of C)   (2)

Hence, number of allowable ways $= (1) - (2) = 24$

[It is also possible to consider only situations of the form ABXXX (or BAXXX) and XABXX (or XBAXX), and multiply by 2 to cover the symmetrical situations where we start from the other end.]